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Classification of three-dimensional evolution algebras

Yolanda Cabrera Casado [a], Mercedes Siles Molina [a,*], M. Victoria Velasco [b]

[a] Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, Campus de Teatinos s/n. 29071 Málaga, Spain
[b] Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

We classify three dimensional evolution algebras over a field having characteristic different from 2 and in which there are roots of orders 2, 3 and 7.

1. Introduction

The use of non-associative algebras to formulate Mendel's laws was started by Etherington in his papers [6,7]. Other genetic algebras (those that model inheritance in

* Corresponding author.
E-mail addresses: yolandacc@uma.es (Y. Cabrera Casado), msilesm@uma.es (M. Siles Molina), vvelasco@ugr.es (M.V. Velasco).

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genetics) called evolution algebras emerged to study non-Mendelian genetics. Its theory in the finite-dimensional case was introduced by Tian in [8]. The systematic study of evolution algebras of arbitrary dimension and of their algebraic properties was started in [1], where the authors analyze evolution subalgebras, ideals, non-degeneracy, simple evolution algebras and irreducible evolution algebras. The aim of this paper is to obtain the classification of three-dimensional evolution algebras having in mind to apply this classification in a near future in a biological setting and to detect possible tools to implement in wider classifications.

Two-dimensional evolution algebras over the complex numbers were determined in [3], although we have found that this classification is incomplete: the algebra A with natural basis $\{e_1, e_2\}$ such that $e_1^2 = e_2$ and $e_2^2 = e_1$ is a two-dimensional evolution algebra not isomorphic to any of the six types in [3]. We realized of this fact when classifying the three-dimensional evolution algebras A such that $\dim(A^2) = 2$ and having annihilator[1] of dimension 1.

The three dimensional case is much more complicated, as can be seen in this work, where we prove that there are 116 types of three-dimensional evolution algebras. The details can be found in [2, Tables 1–24]. Just after finishing this paper we found the article [5], where one of the aims of the authors is to classify indecomposable[2] nilpotent evolution algebras up to dimension five over algebraically closed fields of characteristic not two. The three-dimensional ones can be localized in our classification and for these, it is not necessary to consider algebraically closed fields.

In this paper we deal with evolution algebras over a field $\mathbb{K}$ of characteristic different from 2 and in which every polynomial of the form $x^n - k$, for $n = 2, 3, 7$ and $k \in \mathbb{K}$ has a root in the field. We denote by ϕ a seventh root of the unit and by ζ a third root of the unit.

In Section 2 we introduce the essential definitions. For every arbitrary finite dimensional algebra, fix a basis $B = \{e_i \mid i = 1, \ldots, n\}$. The product of this algebra, relative to the basis B is determined by the matrices of the multiplication operators, $M_B(\lambda_{e_i})$ (see (1)). The relationship under change of basis is also established. In the particular case of evolution algebras Theorem 2.2 shows this connection.

We start Section 3 by analyzing the action of the group $S_3 \rtimes (\mathbb{K}^\times)^3$ on $\mathcal{M}_3(\mathbb{K})$. The orbits of this action will completely determine the non-isomorphic evolution algebras A when $\dim(A^2) = 3$ and in some cases when $\dim(A^2) = 2$.

We have divided our study into four cases depending on the dimension of A^2, which can be 0, 1, 2 or 3. The first case is trivial. The study of the third and of the fourth ones is made by taking into account which are the possible matrices P that appear as change of basis matrices. It happens that for dimension 3, as we have said, the only matrices are those in $S_3 \rtimes (\mathbb{K}^\times)^3$.

[1] The annihilator of A, $\mathrm{ann}(A)$, is defined as the set of those elements x in A such that $xA = 0$.
[2] Irreducible following [1].

When the dimension of A^2 is 2, there exists three groups of cases (four in fact, but two of them are essentially the same). Let $B = \{e_1, e_2, e_3\}$ be a natural basis of A such that $\{e_1^2, e_2^2\}$ is a basis of A^2 and $e_3^2 = c_1 e_1^2 + c_2 e_2^2$ for some $c_1, c_2 \in \mathbb{K}$. The first case happens when $c_1 c_2 \neq 0$. Then, $P \in S_3 \rtimes (\mathbb{K}^\times)^3$. The second group of cases arises when $c_1 = 0$ and $c_2 \neq 0$. Then, the matrix P is id_3, $(2,3)$,[3] or the matrix Q given in Case 2 (when $\dim(A^2 = 2)$). The third one appears when case happens when $c_1, c_2 = 0$. In this case the matrix P is id_3 or the matrices Q' and Q'' given in Case 4 (when $\dim(A^2 = 2)$).

For $P \in S_3 \rtimes (\mathbb{K}^\times)^3$, we classify taking into account: the dimension of the annihilator of A, the number of non-zero entries in the structure matrix (which remains invariant, as it is proved in Proposition 3.2), and if the algebra A satisfies Property (2LI).[4]

For $P \in \{\mathrm{id}_3, (2,3), Q\}$, we obtain a first classification (see the different Figures in [2]). Then we compare which matrices produce isomorphic algebras and eliminating redundancies we get the matrices given in the set S that appears in Theorem 3.5. Again, some of these matrices give isomorphic evolution algebras. In order to classify them, we take into account that the number of non-zero entries of the matrices in S remains invariant under the action of the matrix P (see Remark 3.7). Note that the resulting matrices correspond to evolution algebras with zero annihilator and do not satisfy Property (2LI).

For $P \in \{\mathrm{id}_3, Q', Q''\}$ we classify taking into account that the third column of the structure matrix has three zero entries (the dimension of the annihilator is one and, consequently, they do not satisfy Property (2LI)) and the number of zeros in the first and the second row remains invariant under change of basis matrices (see Remark 3.8).

For $\dim(A^2) = 3$ we classify by the number of non-zero entries in the structure matrix.

In the case $\dim(A^2) = 1$ it is not efficient to tackle the problem of the classification by obtaining the possible change of basis matrices, although for completeness we have determined them in [2, Appendix]. This is because we follow a different pattern. The key point for this study will be the extension property [5] ((EP) for short). We have classified taking into account the following properties: whether or not A^2 has the extension property, the dimension of the annihilator of A, and whether or not the evolution algebra A has a principal[6] two-dimensional evolution ideal which is degenerate[7] as an evolution algebra (PD2EI for short).

The classification of three-dimensional evolution algebras is achieved in Theorem 3.5. We summarize the cases in the tables that follow.

[3] The matrix obtained from the identity matrix, id_3, when exchanging the second and the third rows.

[4] For any basis $\{e_1, e_2, e_3\}$ the ideal A^2 has dimension two and it is generated by $\{e_i^2, e_j^2\}$, for every $i, j \in \{1, 2, 3\}$ with $i \neq j$.

[5] There is a natural basis of A^2 that can be enlarged to a natural basis of A.

[6] Principal means that it is generated as an ideal by one element.

[7] An evolution algebra is non-degenerate if $e^2 \neq 0$ for any element e in any basis (see [1, Definition 2.16 and Corollary 2.19]). Otherwise we say that it is degenerate.

A^2 has EP	dim(ann(A))	A has a PD2EI	Number
No	0	Yes	1
No	1	Yes	1
Yes	2	No	1
Yes	1	No	1
Yes	0	No	1
Yes	2	Yes	1
Yes	1	Yes	1

$\dim(A^2) = 1$

dim(ann(A))	Non-zero entries *Non-zero entries in S **Non-zero entries in rows 1 and 2	A has Property (2LI)	Number
1	1**	No	2
1	2**	No	4
1	3**	No	2
1	4**	No	3
0	4*	No	3
0	5*	No	6
0	6*	No	3
0	7*	No	6
0	8*	No	3
0	9*	No	3
0	4	Yes	4
0	5	Yes	3
0	6	Yes	7
0	7	Yes	6
0	8	Yes	2
0	9	Yes	1

$\dim(A^2) = 2$

Non-zero entries	Number
3	3
4	6
5	16
6	15
7	8
8	2
9	1

$\dim(A^2) = 3$

2. Product and change of basis

In this section we study the product in an arbitrary algebra by considering the matrices associated to the product by any element in a fixed basis. We specialize to the case of evolution algebras and obtain the relationship for two structure matrices of the same evolution algebra relative to different basis.

2.1. The product of an algebra

Let A be a $\mathbb{K}$-algebra. Assume that $B = \{e_i \mid i \in \Lambda\}$ is a basis of A, and let $\{\omega_{kij}\}_{i,j,k\in\Lambda} \subseteq \mathbb{K}$ be the *structure constants*, i.e. $e_ie_j = \sum_{k\in\Lambda} \omega_{kij}e_k$ and ω_{kij} is zero for almost all k. Since in this paper we will deal only with finite dimensional evolution algebras, we will assume that Λ is finite and has cardinal n.

For any element $a \in A$ the following map defines the *left multiplication operator* by a, denoted as λ_a:

$$\lambda_a : \begin{array}{ll} A & \to A \\ x & \mapsto ax \end{array}$$

Then, for every $i \in \Lambda$ we have

$$M_B(\lambda_{e_i}) = \begin{pmatrix} \omega_{1i1} & \cdots & \omega_{1in} \\ \vdots & \ddots & \vdots \\ \omega_{ni1} & \cdots & \omega_{nin} \end{pmatrix},$$

where for any linear map $T : A \to A$ we write $M_B(T)$ to denote the matrix in $\mathcal{M}_\Lambda(\mathbb{K})$ associated to T relative to the basis B.

Let A be an algebra and let $B = \{e_i \mid i \in \Lambda\}$ be a basis of A. For arbitrary elements $x = \sum_{i\in\Lambda} \alpha_i e_i$ and $y = \sum_{i\in\Lambda} \beta_i e_i$ in A the product xy is as follows:

$$xy = \left(\sum_{i\in\Lambda} \alpha_i e_i\right)\left(\sum_{j\in\Lambda} \beta_j e_j\right) = \sum_{i,j\in\Lambda} \alpha_i\beta_j \; e_i e_j = \sum_{i,j\in\Lambda} \left(\alpha_i\beta_j \sum_{k\in\Lambda} \omega_{kij} e_k\right)$$
$$= \sum_{k,i,j\in\Lambda} \alpha_i\beta_j\omega_{kij} e_k.$$

Denote by $\xi_B(x)$ the coordinates of an element x in A relative to the basis B, written by columns. Then:

$$\xi_B(xy) = \xi_B\left(\sum_{k,i,j\in\Lambda} \alpha_i\beta_j\omega_{kij} e_k\right) = \begin{pmatrix} \omega_{111} & \cdots & \omega_{11n} \\ \vdots & \ddots & \vdots \\ \omega_{n11} & \cdots & \omega_{n1n} \end{pmatrix}\begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_1\beta_n \end{pmatrix}$$
$$+ \cdots$$
$$+ \begin{pmatrix} \omega_{1n1} & \cdots & \omega_{1nn} \\ \vdots & \ddots & \vdots \\ \omega_{nn1} & \cdots & \omega_{nnn} \end{pmatrix}\begin{pmatrix} \alpha_n\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{pmatrix}.$$

That is,

$$\xi_B(xy) = \sum_{i\in\Lambda} M_B(\lambda_{e_i}) \begin{pmatrix} \alpha_i\beta_1 \\ \vdots \\ \alpha_i\beta_n \end{pmatrix}. \tag{1}$$

An *evolution algebra* over a field $\mathbb{K}$ is a $\mathbb{K}$-algebra A provided with a basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_ie_j = 0$ whenever $i \neq j$. Such a basis B is called a *natural basis*. Now, the structure constants of A relative to B are the scalars $\omega_{ki} \in \mathbb{K}$ such that $e_i^2 := e_ie_i = \sum_{k\in\Lambda} \omega_{ki}e_k$. The matrix $M_B := (\omega_{ki})$ is said to be the *structure matrix of A relative to B*.

For any finite dimensional evolution algebra A with a natural basis B we have

$$M_B = \sum_{i\in\Lambda} M_B(\lambda_{e_i}).$$

In case of A being an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A, the structure constants satisfy that $\omega_{kij} = 0$ for every $i, j, k \in \Lambda$ with $i \neq j$. If we denote $\omega_{kii} = \omega_{ki}$ we obtain that:

$$\begin{aligned}\xi_B(xy) &= \begin{pmatrix} \omega_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_1\beta_n \end{pmatrix} + \ldots + \begin{pmatrix} 0 & \cdots & 0 & \omega_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_n\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{pmatrix} \\ &= \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{pmatrix},\end{aligned}$$

because for every $i \in \Lambda$ the matrix $M_B(\lambda_{e_i})$ has zero entries except at most in its ith column.

Summarizing,

$$\xi_B(xy) = M_B \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{pmatrix}. \tag{2}$$

Definition 2.1. Let A be an algebra and $B = \{e_i \mid i \in \Lambda\}$ a basis of A. For arbitrary elements $x = \sum_{i\in\Lambda} \alpha_ie_i$ and $y = \sum_{i\in\Lambda} \beta_ie_i$ in A, we define

$$x \bullet_B y = \left(\sum_{i\in\Lambda} \alpha_ie_i\right) \bullet_B \left(\sum_{i\in\Lambda} \beta_ie_i\right) := \sum_{i\in\Lambda} \alpha_i\beta_ie_i.$$

Now, in the case of an evolution algebra we may write (2) as follows.

$$\xi_B(xy) = M_B\left(\xi_B(x) \bullet_B \xi_B(y)\right), \tag{3}$$

where, by abuse of notation, we write $\bullet_B$ to multiply two matrices, by identifying the matrices with the corresponding vectors and multiplying them as in Definition 2.1.

2.2. Change of basis

First, we study the matrix of the product of a finite dimensional arbitrary algebra under change of basis. Then we fix our attention in evolution algebras.

Let $B = \{e_i \mid i \in \Lambda\}$ and $B' = \{f_j \mid j \in \Lambda\}$ be two bases of an algebra A. Suppose that the relation between these bases is given by

$$e_i = \sum_{k\in\Lambda} q_{ki} f_k \quad \text{and} \quad f_i = \sum_{k\in\Lambda} p_{ki} e_k,$$

where $\{p_{ki}\}_{k,i\in\Lambda}$ and $\{q_{ki}\}_{k,i\in\Lambda}$ are subsets of $\mathbb{K}$ such that $P_{BB'} := (q_{ki})$ and $P_{B'B} := (p_{ki})$ are the change of basis matrices. Assume that the structure constants of A relative to B and to B' are, respectively, $\{\varpi_{kij}\}_{i,j,k\in\Lambda}$ and $\{\omega_{kij}\}_{i,j,k\in\Lambda}$. Then, for every $i, j \in \Lambda$:

$$\begin{aligned} f_i f_j &= \left(\sum_{k\in\Lambda} p_{ki} e_k\right)\left(\sum_{t\in\Lambda} p_{tj} e_t\right) = \sum_{k,t\in\Lambda} p_{ki}p_{tj} e_k e_t = \sum_{k,t,m\in\Lambda} p_{ki}p_{tj}\varpi_{mkt} e_m \\ &= \sum_{k,t,m,l\in\Lambda} p_{ki}p_{tj}\varpi_{mkt} q_{lm} f_l = \sum_{l\in\Lambda}\left(\sum_{k,t,m\in\Lambda} (p_{ki}p_{tj}\varpi_{mkt}q_{lm})\right) f_l = \sum_{l\in\Lambda} \omega_{lij} f_l. \end{aligned}$$

Therefore, $\sum_{k,t,m\in\Lambda} (p_{ki}p_{tj}\varpi_{mkt}q_{lm}) = \omega_{lij}$.

Our next aim is to express every ω_{lij} in terms of certain matrices. To find such matrices, write:

$$\begin{aligned} \omega_{lij} = {} & p_{1i}p_{1j}\varpi_{111}q_{l1} + \ldots + p_{1i}p_{1j}\varpi_{n11}q_{ln} \\ & \vdots \\ & + p_{1i}p_{nj}\varpi_{11n}q_{l1} + \ldots + p_{1i}p_{nj}\varpi_{n1n}q_{ln} \\ & \vdots \\ & + p_{ni}p_{1j}\varpi_{1n1}q_{l1} + \ldots + p_{ni}p_{1j}\varpi_{nn1}q_{ln} \\ & \vdots \\ & + p_{ni}p_{nj}\varpi_{1nn}q_{l1} + \ldots + p_{ni}p_{nj}\varpi_{nnn}q_{ln}. \end{aligned}$$

In terms of matrices,

$$\begin{aligned} \omega_{lij} = {} & \begin{pmatrix} q_{l1} & \cdots & q_{ln} \end{pmatrix} \begin{pmatrix} \varpi_{111} & \cdots & \varpi_{11n} \\ \vdots & \ddots & \vdots \\ \varpi_{n11} & \cdots & \varpi_{n1n} \end{pmatrix} \begin{pmatrix} p_{1i}p_{1j} \\ \vdots \\ p_{1i}p_{nj} \end{pmatrix} \\ & + \cdots \end{aligned}$$

$$+ \begin{pmatrix} q_{l1} & \cdots & q_{ln} \end{pmatrix} \begin{pmatrix} \varpi_{1n1} & \cdots & \varpi_{1nn} \\ \vdots & \ddots & \vdots \\ \varpi_{nn1} & \cdots & \varpi_{nnn} \end{pmatrix} \begin{pmatrix} p_{ni}p_{1j} \\ \vdots \\ p_{ni}p_{nj} \end{pmatrix}.$$

This is equivalent to:

$$\begin{aligned} M_{B'}(\lambda_{f_i}) &= \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} \varpi_{111} & \cdots & \varpi_{11n} \\ \vdots & \ddots & \vdots \\ \varpi_{n11} & \cdots & \varpi_{n1n} \end{pmatrix} \begin{pmatrix} p_{1i}p_{11} & \cdots & p_{1i}p_{1n} \\ \vdots & \ddots & \vdots \\ p_{1i}p_{n1} & \cdots & p_{1i}p_{nn} \end{pmatrix} \\ &+ \cdots \\ &+ \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} \varpi_{1n1} & \cdots & \varpi_{1nn} \\ \vdots & \ddots & \vdots \\ \varpi_{nn1} & \cdots & \varpi_{nnn} \end{pmatrix} \begin{pmatrix} p_{ni}p_{11} & \cdots & p_{ni}p_{1n} \\ \vdots & \ddots & \vdots \\ p_{ni}p_{n1} & \cdots & p_{ni}p_{nn} \end{pmatrix} \\ &= P_{B'B}^{-1} \left(\sum_k M_B(\lambda_{e_k}) p_{ki} \right) P_{B'B}. \end{aligned}$$

We finish the section by asserting the relationship among two structure matrices associated to the same evolution algebra relative to different bases. We include the proof of Theorem 2.2 for completeness. The ideas we have used can be found in [8, Section 3.2.2.].

Theorem 2.2. *Let A be an evolution algebra and let $B = \{e_1, \dots, e_n\}$ be a natural basis of A with structure matrix $M_B = (\omega_{ij})$. Then:*

(i) *If $B' = \{f_1, \dots, f_n\}$ is a natural basis of A and $P = (p_{ij})$ is the change of basis matrix $P_{B'B}$, i.e., $f_i = \sum_j p_{ji} e_j$, for every i, then $|P| \neq 0$ and*

$$\begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \left[\begin{pmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{pmatrix} \bullet_B \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{for every } i \neq j. \tag{4}$$

Moreover,

$$M_{B'} = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}^{-1} \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} p_{11}^2 & \cdots & p_{1n}^2 \\ \vdots & \ddots & \vdots \\ p_{n1}^2 & \cdots & p_{nn}^2 \end{pmatrix} = P^{-1} M_B P^{(2)}, \tag{5}$$

where $P^{(2)} = (p_{ij}^2)$.

(ii) *Assume that* $P = (p_{ij}) \in \mathcal{M}_n(\mathbb{K})$ *has non-zero determinant and satisfies the relations in* (4). *Define* $B' = \{f_1, \ldots, f_n\}$, *where* $f_i = \sum_j p_{ji} e_j$, *for every* i. *Then,* B' *is a natural basis and* (5) *is satisfied.*

Proof. (i). Clearly, since B and B' are two bases of A then $|P| \neq 0$. Besides, since B and B' are natural bases, by (2) we have:

$$\xi_B(f_i f_j) = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \left[\begin{pmatrix} p_{1i} \\ \vdots \\ p_{ni} \end{pmatrix} \bullet_B \begin{pmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\xi_B(f_i^2) = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} p_{1i}^2 \\ \vdots \\ p_{ni}^2 \end{pmatrix}$$

for every i, j, being $i \neq j$.

On the other hand, if $M_{B'} = (\varpi_{ij})$, for every $i \neq j$ we obtain:

$$\xi_{B'}(f_i^2) = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}^{-1} \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} p_{1i}^2 \\ \vdots \\ p_{ni}^2 \end{pmatrix} = \begin{pmatrix} \varpi_{1i} \\ \vdots \\ \varpi_{ni} \end{pmatrix}$$

and consequently

$$M_{B'} = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}^{-1} \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} p_{11}^2 & \cdots & p_{1n}^2 \\ \vdots & \ddots & \vdots \\ p_{n1}^2 & \cdots & p_{nn}^2 \end{pmatrix} = P^{-1} M_B P^{(2)}.$$

(ii). Assume that $P = (p_{ij})$ has non zero determinant. Then B', defined as in the statement, is a basis of A. Moreover, if (4) is satisfied, then B' is a natural basis as follows by (2). $\square$

The formula (4) can be rewritten in a more condensed way. Concretely (see [8]),

$$M_B(P * P) = 0, \tag{6}$$

where $P * P = (c_{k(i,j)}) \in \mathcal{M}_{n \times \frac{n(n-1)}{2}}(\mathbb{K})$, being $c_{k(i,j)} = p_{ki} p_{kj}$ for every pair (i, j) with $i < j$ and $i, j \in \{1, \ldots, n\}$.

3. Three-dimensional evolution algebras

The aim of this section is to determine the three-dimensional evolution algebras over a field $\mathbb{K}$ having characteristic different from two and such that for any $\alpha \in \mathbb{K}$ and $n = 2, 3, 7$, the equation $x^n = \alpha$ has a solution. For our purposes, we divide our study in different cases, depending on the dimension of A^2.

3.1. Action of $S_3 \rtimes (\mathbb{K}^\times)^3$ on $\mathcal{M}_3(\mathbb{K})$

Let $\mathbb{K}$ be a field. By $\mathbb{K}^\times$ we denote $\mathbb{K} \setminus \{0\}$. For every $\alpha, \beta, \gamma \in \mathbb{K}^\times$, we define the matrices:

$$\Pi_1(\alpha) := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Pi_2(\beta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Pi_3(\gamma) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

It is easy to prove that they commute each other. This implies that

$$G = \left\{\Pi_1(\alpha)\Pi_2(\beta)\Pi_3(\gamma) \mid \alpha, \beta, \gamma \in \mathbb{K}^\times\right\} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{K}^\times \right\}$$

is an abelian subgroup of $\mathrm{GL}_3(\mathbb{K})$. We will denote the diagonal matrix $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ by (α, β, γ). With this notation in mind, it is immediate to see that $G \cong \mathbb{K}^\times \times \mathbb{K}^\times \times \mathbb{K}^\times$ with product given by $(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') := (\alpha\alpha', \beta\beta', \gamma\gamma')$.

Now, consider the symmetric group S_3 of all permutations of the set $\{1, 2, 3\}$. The standard notation for S_3 is:

$$S_3 = \{id, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\},$$

where id is the identity map, (i, j) is the permutation that sends the element i into the element j and (i, j, k) is the permutation sending i to j, j to k and k to i, for $\{i, j, k\} = \{1, 2, 3\}$.

We may identify S_3 with the set

$$\left\{ \mathrm{id}_3, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} \tag{7}$$

in the following way: id is identified with the identity matrix id_3, $(1, 2)$ with the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

because this matrix appears when permuting the first and the second columns of id_3, etc. The matrices in (7) are called 3×3 *permutation matrices*.

From now on, we will consider that S_3 consists of the permutation matrices.

This allows to see S_3 as a subgroup of $\mathrm{GL}_3(\mathbb{K})$. Denote by H the subgroup of $\mathrm{GL}_3(\mathbb{K})$ generated by S_3 and $(\mathbb{K}^\times)^3$.

It is not difficult to verify that for every $\sigma \in S_3$ and every $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{K}^\times)^3$ its product is as follows:

$$(\lambda_1, \lambda_2, \lambda_3)\sigma = \sigma(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}).$$

Therefore, we may write

$$H = \{\sigma(\alpha, \beta, \gamma) \mid \sigma \in S_3,\ (\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3\}.$$

The multiplication in H is given by

$$\begin{aligned} \sigma(\alpha_1, \alpha_2, \alpha_3)\tau(\beta_1, \beta_2, \beta_3) &= \sigma\tau(\alpha_{\tau(1)}, \alpha_{\tau(2)}, \alpha_{\tau(3)})(\beta_1, \beta_2, \beta_3) \\ &= \sigma\tau(\alpha_{\tau(1)}\beta_1, \alpha_{\tau(2)}\beta_2, \alpha_{\tau(3)}\beta_3). \end{aligned} \tag{8}$$

A *semidirect product* of S_3 and $(\mathbb{K}^\times)^3$ is defined as $S_3 \times (\mathbb{K}^\times)^3$ with product as in (8). It is denoted by

$$S_3 \rtimes (\mathbb{K}^\times)^3.$$

Notice that $S_3 \rtimes (\mathbb{K}^\times)^3$ coincides with

$$\left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha \\ 0 & \beta & 0 \\ \gamma & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{K}^\times \right\} \tag{9}$$

Thus, $S_3 \rtimes (\mathbb{K}^\times)^3 = \{(\alpha, \beta, \gamma)\sigma \mid \alpha, \beta, \gamma \in \mathbb{K}^\times,\ \ \sigma \in S_3\}$.

We define the action of $S_3 \rtimes (\mathbb{K}^\times)^3$ on the set $\mathcal{M}_3(\mathbb{K})$ given by:

$$\sigma \cdot \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} := \begin{pmatrix} \omega_{\sigma(1)\sigma(1)} & \omega_{\sigma(1)\sigma(2)} & \omega_{\sigma(1)\sigma(3)} \\ \omega_{\sigma(2)\sigma(1)} & \omega_{\sigma(2)\sigma(2)} & \omega_{\sigma(2)\sigma(3)} \\ \omega_{\sigma(3)\sigma(1)} & \omega_{\sigma(3)\sigma(2)} & \omega_{\sigma(3)\sigma(3)} \end{pmatrix}. \tag{10}$$

$$(\alpha, \beta, \gamma) \cdot \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} := \begin{pmatrix} \alpha\omega_{11} & \frac{\beta^2}{\alpha}\omega_{12} & \frac{\gamma^2}{\alpha}\omega_{13} \\ \frac{\alpha^2}{\beta}\omega_{21} & \beta\omega_{22} & \frac{\gamma^2}{\beta}\omega_{23} \\ \frac{\alpha^2}{\gamma}\omega_{31} & \frac{\beta^2}{\gamma}\omega_{32} & \gamma\omega_{33} \end{pmatrix} \tag{11}$$

for every $\sigma \in S_3$ and every $(\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3$.

For arbitrary $P \in S_3 \rtimes (\mathbb{K}^\times)^3$ and $M \in \mathcal{M}_3(\mathbb{K})$, the action of P on M can be formulated as follows:

$$P \cdot M := P^{-1} M P^{(2)}. \tag{12}$$

Remark 3.1. The action given in (12) has been inspired by Condition (5) in Theorem 2.2. Notice that any matrix P in $S_3 \rtimes (\mathbb{K}^\times)^3$ is a change of basis matrix from a natural basis B into another natural basis B' and the relationship among the structure matrices M_B and $M_{B'}$ and the matrix P is as given in Condition (5), that is, $P^{-1} M_B P^{(2)} = M'_B$. This is the reason because we define the action of P on M_B by:

$$P \cdot M_B = P^{-1} M_B P^{(2)}.$$

The result that follows will be very useful in Theorem 3.5.

Proposition 3.2. *For any $P \in S_3 \rtimes (\mathbb{K}^\times)^3$ and any $M \in \mathcal{M}_3(\mathbb{K})$ we have:*

(i) *The number of zero entries in M coincides with the number of zero entries in $P \cdot M$.*
(ii) *The number of zero entries in the main diagonal of M coincides with the number of zero entries in the main diagonal of $P \cdot M$.*
(iii) *The rank of M and the rank of $P \cdot M$ coincide.*
(iv) *Assume that M is the structure matrix of an evolution algebra A relative to a natural basis B. Assume that $A^2 = A$. If N is the structure matrix of A relative to a natural basis B' then there exists $Q \in S_3 \rtimes (\mathbb{K}^\times)^3$ such that $N = Q \cdot M$.*

Proof. Fix an element P in $S_3 \rtimes (\mathbb{K}^\times)^3$. Then there exist $\sigma \in S_3$ and $(\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3$ such that $P = \sigma(\alpha, \beta, \gamma)$. Therefore $P \cdot M = (\sigma(\alpha, \beta, \gamma)) \cdot M = \sigma \cdot ((\alpha, \beta, \gamma) \cdot M)$. Item (i) and (ii) follows by (10) and (11). Item (iii) is easy to show because $P \cdot M = P^{-1} M P^{(2)}$ and P is an invertible matrix. Finally, (iv) follows from the definition of the action and [4, Theorem 4.4]. □

3.2. Main theorem

Here we prove the main result of the paper: the classification of three-dimensional evolution algebras over a field of characteristic different from two in which there are roots of orders two, three and seven.

Definitions 3.3. (See [1, Definitions 2.4].) An *evolution subalgebra* of an evolution algebra A is a subalgebra $A' \subseteq A$ such that A' is an evolution algebra, i.e. A' has a natural basis.

We say that A' has the *extension property* if there exists a natural basis B' of A' which can be extended to a natural basis of A.

An evolution algebra is *non-degenerate* if $e^2 \neq 0$ for any element e in any basis (see [1, Definition 2.16 and Corollary 2.19]). Otherwise we say that it is *degenerate*. Note that this definition does not depend on the basis as proved in [1].

Definition 3.4. A three dimensional evolution algebra A is said to have *Property* (2LI) if for any basis $\{e_1, e_2, e_3\}$ of A, the ideal A^2 has dimension two and it is generated by $\{e_i^2, e_j^2\}$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Theorem 3.5. *Let A be a three-dimensional evolution $\mathbb{K}$-algebra.*

(i) *If $\dim(A^2) = 0$ then $M_B = 0$ for any natural basis B of A.*

(ii) *If $\dim(A^2) = 1$ then there exists a natural basis B such that M_B is one of the seven matrices given in Table 1. All of them produce mutually non-isomorphic evolution algebras. The algebras in this case are completely classified by the following properties: having A^2 the extension property, $\dim(\mathrm{ann}(A))$, and whether or not A has a principal ideal of dimension two which is degenerate.*

(iii) *If $\dim(A^2) = 2$, then there exists a natural basis B such that M_B is one of the matrices given in the Cases 1 to 4. There are 57 possible cases. Let $B = \{e_1, e_2, e_3\}$ be such that $\{e_1^2, e_2^2\}$ is a basis of A^2 and $e_3^2 = c_1 e_1^2 + c_2 e_2^2$, for $c_1, c_2 \in \mathbb{K}$.*

   (a) *If $c_1 c_2 \neq 0$, then $\dim(\mathrm{ann}(A)) = 0$; the algebra A has Property (2LI) and the number of non-zero entries in M_B can be 4 to 9.*

   (b) *If $c_1 = 0$ and $c_2 \neq 0$ (the case $c_2 = 0$ and $c_1 \neq 0$ is analogous), then the evolution algebras appearing have $\dim(\mathrm{ann}(A)) = 0$; the algebra A has not Property (2LI) and the number of non-zero entries in the set that follows can be from 4 to 9:*

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \lambda & \lambda \end{pmatrix} \right\}.$$

   (c) *If $c_1, c_2 = 0$, then the evolution algebras appearing have $\dim(\mathrm{ann}(A)) = 1$; the algebra A has not Property (2LI) and the number of non-zero entries in rows one and two can be from 1 to 4.*

(iv) *If $\dim(A^2) = 3$ then there exists a natural basis B such that M_B is one of the matrices given in the Cases 1 to 7. They are completely determined by the number of non-zero entries in M_B. There are 51 possible cases.*

Proof. Fix a three-dimensional evolution algebra A and a natural basis $B = \{e_1, e_2, e_3\}$. Let M_B be the structure matrix of A relative to B:

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}.$$

In order to classify all the three dimensional evolution algebras we try to find a basis of A for which its structure matrix has an expression as easy as possible, where by 'easy' we mean with the maximum number of 0, 1 and -1 in the entries.

Case $\dim(A^2) = 0$**.**
Then $M_B = 0$ and there is a unique evolution algebra.

Case $\dim(A^2) = 1$**.**
Without loss in generality we may assume $e_1^2 \neq 0$. Write $e_1^2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$, where $\omega_i \in \mathbb{K}$ and $\omega_i \neq 0$ for some i. Note that $\{e_1^2\}$ is a basis of A^2.

Since $e_2^2, e_3^2 \in A^2$, there exist $c_1, c_2 \in \mathbb{K}$ such that

$$e_2^2 = c_1 e_1^2 = c_1(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3),$$
$$e_3^2 = c_2 e_1^2 = c_2(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3).$$

Then

$$M_B = \begin{pmatrix} \omega_1 & c_1\omega_1 & c_2\omega_1 \\ \omega_2 & c_1\omega_2 & c_2\omega_2 \\ \omega_3 & c_1\omega_3 & c_2\omega_3 \end{pmatrix}.$$

We start the study of this case by paying attention to the algebraic properties of the evolution algebras that we consider. To see which are the matrices that appear as change of basis matrices, we refer the reader to Appendix in [2].

We analyze when A^2 has the extension property. That is, if there exists a natural basis $B' = \{e_1', e_2', e_3'\}$ of A with

$$e_1' = e_1^2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \tag{13}$$
$$e_2' = \alpha e_1 + \beta e_2 + \gamma e_3$$
$$e_3' = \delta e_1 + \nu e_2 + \eta e_3,$$

for some $\alpha, \beta, \gamma, \delta, \nu, \eta \in \mathbb{K}$ that we may choose satisfying $\nu(\beta - \gamma) \neq 0$. Being B' a basis implies

$$\mid P_{B'B} \mid = \begin{vmatrix} \omega_1 & \alpha & \delta \\ \omega_2 & \beta & \nu \\ \omega_3 & \gamma & \eta \end{vmatrix} \neq 0. \tag{14}$$

By Theorem 2.2, B' is a natural basis if and only if the following conditions are satisfied.

$$\alpha\omega_1 + \beta\omega_2 c_1 + \gamma\omega_3 c_2 = 0 \tag{15}$$
$$\delta\omega_1 + \nu\omega_2 c_1 + \eta\omega_3 c_2 = 0 \tag{16}$$
$$\alpha\delta + \beta\nu c_1 + \gamma\eta c_2 = 0$$

In these conditions, the structure matrix of A relative to B' is:

$$M_{B'} = \begin{pmatrix} \omega_1^2 + \omega_2^2 c_1 + \omega_3^2 c_2 & \alpha^2 + \beta^2 c_1 + \gamma^2 c_2 & \delta^2 + \nu^2 c_1 + \eta^2 c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the computations, we will take into account (11). On the other hand, to find the different mutually non-isomorphic evolution algebras it will be very useful to study if they have a two-dimensional evolution ideal generated by one element which is degenerate as an evolution algebra.

Now, we start with the analysis of the different cases.

Case 1. Suppose that $\omega_1 \neq 0$.
By changing the basis, we may consider $e_1^2 = e_1 + \omega_2 e_2 + \omega_3 e_3$. Using (15) we get $\alpha = -(\beta\omega_2 c_1 + \gamma\omega_3 c_2)$ and by (16), $\delta = -(\nu\omega_2 c_1 + \eta\omega_3 c_2)$. If we replace α and δ in (14) we obtain that:

$$\mid P_{B'B} \mid = (1 + \omega_2^2 c_1 + \omega_3^2 c_2)\nu(\beta - \gamma).$$

Now we distinguish if $\mid P_{B'B} \mid$ is zero or not. This happens depending on $1+\omega_2^2 c_1+\omega_3^2 c_2$ being zero or not.

Case 1.1 Assume $1 + \omega_2^2 c_1 + \omega_3^2 c_2 = 0$.
In this case A^2 has not the extension property since $|P_{B'B}| = 0$. We will analyze what happens when $1 + \omega_3^2 c_2 \neq 0$ and when $1 + \omega_3^2 c_2 = 0$.

Case 1.1.1 If $1 + \omega_3^2 c_2 \neq 0$.
Note that $\omega_2^2 c_1 \neq 0$ since otherwise we get a contradiction. Then $c_1 = \dfrac{-1 - \omega_3^2 c_2}{\omega_2^2}$. In this case, the structure matrix is:

$$M_B = \begin{pmatrix} 1 & \dfrac{-1 - \omega_3^2 c_2}{\omega_2^2} & c_2 \\ \omega_2 & \dfrac{-1 - \omega_3^2 c_2}{\omega_2} & c_2\omega_2 \\ \omega_3 & \dfrac{(-1 - \omega_3^2 c_2)\omega_3}{\omega_2^2} & c_2\omega_3 \end{pmatrix}.$$

Case 1.1.1.1 Suppose that $\omega_3 \neq 0$.
If we take the natural basis $B'' = \{e_1, \omega_2 e_2, \omega_3 e_3\}$, then

$$M_{B''} = \begin{pmatrix} 1 & -1-\omega_3^2 c_2 & \omega_3^2 c_2 \\ 1 & -1-\omega_3^2 c_2 & \omega_3^2 c_2 \\ 1 & -1-\omega_3^2 c_2 & \omega_3^2 c_2 \end{pmatrix}. \tag{17}$$

We are going to distinguish two cases: $c_2 = 0$ and $c_2 \neq 0$.

Assume first $c_2 = 0$. Then $M_{B''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$. By considering another change of basis we find a structure matrix with more zeros. Concretely, let $B''' = \{e_2, e_1 + e_3, e_3\}$. Then

$$M_{B'''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows we will assume that $c_2 \neq 0$. We recall that we are considering the structure matrix given in (17). Take $I := < (1 + \omega_3^2 c_2)e_1 + e_2 >$. Then I is a two-dimensional evolution ideal which is degenerate as an evolution algebra.

Now, for B''' the natural basis given by

$$P_{B'''B''} = \begin{pmatrix} \frac{1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & \frac{-1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & -(\omega_3^2c_2) \\ \frac{-1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & \frac{1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & 0 \\ \frac{1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & \frac{-1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & 1 \end{pmatrix}$$

we obtain:

$$M_{B'''} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $|P_{B'''B''}| = -2(\omega_3^2 c_2)(1 + \omega_3^2 c_2)^2 \neq 0$ because $\omega_3^2 c_2 \neq 0$ and $\omega_3^2 c_2 \neq -1$.

Case 1.1.1.2 Suppose that $\omega_3 = 0$.
Then $1 + \omega_2^2 c_1 = 0$ and necessarily $\omega_2^2 c_1 \neq 0$. In this case,

$$M_B = \begin{pmatrix} 1 & \frac{-1}{\omega_2^2} & c_2 \\ \omega_2 & \frac{-1}{\omega_2} & c_2\omega_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

Again we will distinguish two cases depending on c_2.

Assume $c_2 \neq 0$. Take $B'' = \{e_1, \omega_2 e_2, \frac{1}{\sqrt{c_2}} e_3\}$. Then $M_{B''} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which has already appeared.

Suppose $c_2 = 0$. Then, for $B'' = \{e_1, \omega_2 e_2, e_3\}$ we have $M_{B''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, matrix that has already appeared.

Case 1.1.2 Suppose that $1 + \omega_3^2 c_2 = 0$.
This implies that $\omega_3^2 c_2 \neq 0$ and $\omega_2^2 c_1 = 0$.

Case 1.1.2.1 Assume $c_1 \neq 0$.
This implies that $\omega_2 = 0$. Moreover, as $\omega_3 \neq 0$, necessarily $c_2 = \frac{-1}{\omega_3^2}$. If we take the natural basis $B'' = \{e_1, e_3, e_2\}$, then $M_{B''} = \begin{pmatrix} 1 & \frac{-1}{\omega_3^2} & c_1 \\ \omega_3 & \frac{-1}{\omega_3} & \omega_3 c_1 \\ 0 & 0 & 0 \end{pmatrix}$ and we are as in Case 1.1.1.2.

Case 1.1.2.2 Suppose $c_1 = 0$ and $\omega_2 = 0$.
Take $B'' = \{e_1, \omega_3 e_3, e_2\}$. Then $M_{B''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ again.

Case 1.1.2.3 Assume $c_1 = 0$ and $\omega_2 \neq 0$.
Taking $B'' = \{e_1, e_3, e_2\}$, we are in the same conditions as in Case 1.1.1.1 with $c_2 = 0$.

Case 1.2 Assume $1 + \omega_2^2 c_1 + \omega_3^2 c_2 \neq 0$.
We will prove that A^2 has the extension property. In any subcase we will provide with a natural basis for A one of which elements constitutes a natural basis of A^2.

Case 1.2.1 Suppose that $c_1 = c_2 = 0$.
Consider the natural basis $B' = \{e_1^2, e_2 + e_3, 2e_2 + e_3\}$. Then

$$M_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that this evolution algebra does not have a two-dimensional evolution ideal generated by one element. To prove this, consider $f = me_1 + ne_2 + pe_3$. Then the ideal I that it generates is the linear span of $\{f\} \cup \{m^i e_1\}_{i \in \mathbb{N}}$. In order for I to have a natural basis with two elements, necessarily $m = 0$, implying that the dimension of I is one, a contradiction.

Case 1.2.2 Assume that $c_1 = 0$ and $c_2 \neq 0$.
Then $1 + c_2\omega_3^2 \neq 0$. For $B' = \{e_1 + \omega_2 e_2 + \omega_3 e_3, e_2, -\omega_3 c_2 e_1 + e_2 + e_3\}$ the structure matrix is

$$M_{B'} = \begin{pmatrix} 1 + c_2\omega_3^2 & 0 & c_2(1 + c_2\omega_3^2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that A^2 has the extension property because the first element in B' is e_1^2, which is a natural basis of A^2.

Consider $B'' = \left\{\frac{1}{1+c_2\omega_3^2}\ e_1, e_2, \frac{1}{\sqrt{c_2}(1+c_2\omega_3^2)}\ e_3\right\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element. Let $f = me_1 + ne_2 + pe_3$. Then the ideal generated by f, say I, is the linear span of $\{f, pe_1, me_1\} \cup \{(m^2+p^2)m^i e_1\}_{i\in\mathbb{N}\cup\{0\}} \cup \{(m^2+p^2)^2 m^i e_1\}_{i\in\mathbb{N}\cup\{0\}}$. After some computations, in order for I to have dimension 2 and to be degenerated we get $m = 0$ or $p = 0$, a contradiction.

Case 1.2.3 If $c_1 \neq 0$ and $c_2 \neq 0$.

Case 1.2.3.1 Assume $1 + \omega_2^2 c_1 \neq 0$.

For B' the natural basis such that $P_{B'B} = \begin{pmatrix} 1 & -\omega_2 c_1 & \frac{-\omega_3 c_2}{1+c_1\omega_2^2} \\ \omega_2 & 1 & \frac{-\omega_3\omega_2 c_2}{1+c_1\omega_2^2} \\ \omega_3 & 0 & 1 \end{pmatrix}$, we obtain that $M_{B'} = \begin{pmatrix} 1+\omega_2^2 c_1+\omega_3^2 c_2 & c_1(1+c_1\omega_2^2) & \frac{c_2\left(1+\omega_2^2 c_1+\omega_3^2 c_2\right)}{\left(1+c_1\omega_2^2\right)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+\omega_2^2 c_1+\omega_3^2 c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1\omega_2^2)(1+\omega_2^2 c_1+\omega_3^2 c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1\omega_2^2}}{\sqrt{c_2}(1+c_1\omega_2^2+c_2\omega_3^2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is not difficult to show that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element.

Case 1.2.3.2 Assume $1 + \omega_2^2 c_1 = 0$.

Then $\omega_2\omega_3 c_1 c_2 \neq 0$ and so $c_1 = -1/\omega_2^2$. For B' such that $P_{B'B} = \begin{pmatrix} 1 & \frac{-c_2}{2} & \frac{2-\omega_3^2 c_2}{2} \\ \omega_2 & \frac{\omega_2 c_2}{2} & \omega_2(1+\frac{1}{2}c_2\omega_3^2) \\ \omega_3 & \frac{1}{\omega_3} & \omega_3 \end{pmatrix}$ we have $M_{B'} = \begin{pmatrix} \omega_3^2 c_2 & \frac{c_2}{\omega_3^2} & -\omega_3^2 c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Now, we consider the natural basis B'' for which $P_{B''B'} = \begin{pmatrix} \frac{1}{\omega_3^2 c_2} & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \\ 0 & 0 & \frac{\sqrt{-1}}{\omega_3^2 c_2} \end{pmatrix}$. Then, the structure matrix is $M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.4 Suppose that $c_1 \neq 0$ and $c_2 = 0$.
Considering the natural basis $B'' = \{e_1, e_3, e_2\}$ we obtain $M_{B''} = \begin{pmatrix} 1 & 0 & c_1 \\ \omega_3 & 0 & \omega_3 c_1 \\ \omega_2 & 0 & \omega_2 c_1 \end{pmatrix}$, and we are in the same conditions as in Case 1.2.1.2.

Case 2 Suppose that $\omega_1 = 0$.
The structure matrix of the evolution algebra is

$$M_B = \begin{pmatrix} 0 & 0 & 0 \\ \omega_2 & \omega_2 c_1 & \omega_2 c_2 \\ \omega_3 & \omega_3 c_1 & \omega_3 c_2 \end{pmatrix}.$$

Necessarily there exists $i \in \{2, 3\}$ such that $\omega_i \neq 0$. Without loss in generality we assume $\omega_2 \neq 0$.

Case 2.1 Assume $c_1 \neq 0$.
Consider the natural basis $B'' = \{e_2, e_3, e_1\}$. Then $M_{B''} = \begin{pmatrix} \omega_2 c_1 & \omega_2 c_2 & 1 \\ \omega_3 c_1 & \omega_3 c_2 & \omega_3 \\ 0 & 0 & 0 \end{pmatrix}$ and we are in the same conditions as in Case 1.

Case 2.2 If $c_1 = 0$.

Case 2.2.1 Assume $c_2\omega_3 \neq 0$.
Taking the natural basis $B'' = \{e_3, e_2, e_1\}$, then $M_{B''} = \begin{pmatrix} \omega_3 c_2 & 0 & \omega_3 \\ \omega_2 c_2 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix}$ and we are in the same conditions as in Case 1.

Case 2.2.2 Suppose that $c_2\omega_3 = 0$.

Case 2.2.2.1 Assume $c_2 = 0$.
Take the natural basis $B' = \{\omega_2 e_2 + \omega_3 e_3, \frac{1}{\omega_2} e_3, e_1\}$. Then

$$M_{B'} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that A^2 has the extension property.

Case 2.2.2.2 Assume $c_2 \neq 0$.
Then $\omega_3 = 0$. For $B' = \{\omega_2 e_2, e_1, \frac{1}{\sqrt{c_2}} e_3\}$ we have

$$M_{B'} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, A^2 has also the extension property.

We have completed the study of all the cases and will list them in Table 1. All of them produces evolution algebras A such that $\dim(A) = 3$ and $\dim(A^2) = 1$. They are mutually non-isomorphic, as will be clear from the table. We specify the following

Table 1
$\dim(A^2) = 1$.

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_3 >$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	$I =< e_3 >$

properties that are invariant under isomorphisms of evolution algebras: Whether or not A^2 has the extension property, the dimension of the annihilator of A, and whether or not A has a principal degenerate two-dimensional evolution ideal.

In every of the cases listed in Table 1 we have analyzed when A^2 has the extension property. To compute the dimension of the annihilator we have used [1, Proposition 2.18]. We also specify in the table if the evolution algebra has or not a two-dimensional evolution ideal, which is degenerate as an evolution algebra, and which is generated by one element.

Recall that for a commutative algebra A the *annihilator* of A, denoted by $\mathrm{ann}(A)$ is defined to be $\mathrm{ann}(A) = \{x \in A \mid xA = 0\}$.

Case $\dim(A^2) = 2$**.**
The first step is to compute the possible matrices $P_{B'B}$ for natural basis B and B'. Without loss in generality, we may assume that there exists a natural basis $B = \{e_1, e_2, e_3\}$ such that

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_1\omega_{11} + c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_1\omega_{21} + c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_1\omega_{31} + c_2\omega_{32} \end{pmatrix} \tag{19}$$

for some $c_1, c_2 \in \mathbb{K}$ with $\omega_{11}\omega_{22} - \omega_{12}\omega_{21} \neq 0$.

Let B' be another natural basis and let $P_{B'B}$ be the change of basis matrix. Write

$$P_{B'B} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Since B' is a natural basis, by (6) it verifies:

$$\begin{cases} \omega_{11}p_{11}p_{12} + \omega_{12}p_{21}p_{22} + (\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} = 0 \\ \omega_{21}p_{11}p_{12} + \omega_{22}p_{21}p_{22} + (\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} = 0 \\ \omega_{31}p_{11}p_{12} + \omega_{32}p_{21}p_{22} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{32} = 0 \end{cases} \tag{20}$$

$$\begin{cases} \omega_{11}p_{11}p_{13} + \omega_{12}p_{21}p_{23} + (\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{33} = 0 \\ \omega_{21}p_{11}p_{13} + \omega_{22}p_{21}p_{23} + (\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{33} = 0 \\ \omega_{31}p_{11}p_{13} + \omega_{32}p_{21}p_{23} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{33} = 0 \end{cases} \tag{21}$$

$$\begin{cases} \omega_{11}p_{12}p_{13} + \omega_{12}p_{22}p_{23} + (\omega_{11}c_1 + \omega_{12}c_2)p_{32}p_{33} = 0 \\ \omega_{21}p_{12}p_{13} + \omega_{22}p_{22}p_{23} + (\omega_{21}c_1 + \omega_{22}c_2)p_{32}p_{33} = 0 \\ \omega_{31}p_{12}p_{13} + \omega_{32}p_{22}p_{23} + (\omega_{31}c_1 + \omega_{32}c_2)p_{32}p_{33} = 0 \end{cases} \tag{22}$$

We consider the homogeneous system (20) in the three variables $p_{11}p_{12}$, $p_{21}p_{22}$ and $p_{31}p_{32}$. Taking into account that the rank of this system is 2, we may compute its solutions as follows:

$$p_{11}p_{12} = \frac{\begin{vmatrix} -(\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} & \omega_{12} \\ -(\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} & \omega_{22} \end{vmatrix}}{\omega_{11}\omega_{22} - \omega_{21}\omega_{12}} = -c_1p_{31}p_{32} \tag{23}$$

$$p_{21}p_{22} = \frac{\begin{vmatrix} \omega_{11} & -(\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} \\ \omega_{21} & -(\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} \end{vmatrix}}{\omega_{11}\omega_{22} - \omega_{21}\omega_{12}} = -c_2p_{31}p_{32} \tag{24}$$

In an analogous way, we may consider the systems given in (21) and (22). Their solutions can be computed as follows:

$$p_{11}p_{13} = -c_1p_{31}p_{33}; \quad p_{21}p_{23} = -c_2p_{31}p_{33}; \tag{25}$$

and

$$p_{12}p_{13} = -c_1p_{32}p_{33}; \quad p_{22}p_{23} = -c_2p_{32}p_{33}. \tag{26}$$

Case 1 $c_1c_2 \neq 0$.

In this case the annihilator is zero because there cannot be a column of zeros (apply [1, Proposition 2.18]). All the evolution algebras appearing in this case will have Property (2LI), that we define.

Definition 3.6. A three-dimensional evolution algebra A is said to have *Property* (2LI) if for any basis $\{e_1, e_2, e_3\}$ the set $\{e_i^2, e_j^2\}$ is linearly independent, for every $i, j \in \{1, 2, 3\}$ with $i \neq j$.

In what follows we prove, by way of contradiction, that $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$. Note that (see (9)) elements in $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$ are those invertible matrices in $\mathcal{M}_3(\mathbb{K})$ having two zeros in every row and every column.

Then, let $P_{B'B} \notin S_3 \rtimes (\mathbb{K}^\times)^3$. Assume, for example, that $p_{31}p_{32}p_{33} \neq 0$. By (25) and (26) we have that $p_{11}p_{12}p_{13} \neq 0$ and $p_{21}p_{22}p_{23} \neq 0$. If we replace $p_{11} = \dfrac{-c_1p_{31}p_{32}}{p_{12}}$ and $p_{21} = \dfrac{-c_2p_{31}p_{32}}{p_{22}}$ in (25) we obtain $p_{13} = \dfrac{p_{33}p_{12}}{p_{32}}$ and $p_{23} = \dfrac{p_{33}p_{22}}{p_{32}}$. Finally, if we replace these two values in (26), we get $p_{12}^2 = -c_1p_{32}^2$ and $p_{22}^2 = -c_2p_{32}^2$. Therefore,

$$p_{12} = \pm\sqrt{-c_1}\, p_{32}$$
$$p_{22} = \pm\sqrt{-c_2}\, p_{32}$$

and

$$p_{13} = \pm\sqrt{-c_1}\, p_{33}$$
$$p_{23} = \pm\sqrt{-c_2}\, p_{33}$$
$$p_{21} = \pm\sqrt{-c_2}\, p_{31}$$
$$p_{11} = \pm\sqrt{-c_1}\, p_{31}$$

Now,

$$|P_{B'B}| = p_{11}p_{22}p_{33} + p_{12}p_{23}p_{31} + p_{13}p_{21}p_{32} - p_{13}p_{22}p_{31} - p_{21}p_{12}p_{33} - p_{11}p_{32}p_{23} = 0.$$

This is a contradiction. Therefore $p_{31}p_{32}p_{33} = 0$, hence, there exists at least one $i \in \{1, 2, 3\}$ such that $p_{3i} = 0$. We may suppose without loss in generality that $p_{31} = 0$. This means that $p_{11}p_{12} = 0$, $p_{21}p_{22} = 0$, $p_{11}p_{13} = 0$, $p_{21}p_{23} = 0$ and, obviously, $p_{31}p_{32} = 0$ and $p_{31}p_{33} = 0$.

We claim that $P_{B'B}$ has two zero entries in every row and column. In other words, that $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.

Assume $p_{11} = 0$. Since $|P_{B'B}| \neq 0$, necessarily $p_{21} \neq 0$. So, $p_{23} = p_{22} = 0$ and consequently, using (26), $p_{32}p_{33} = 0$ and $p_{12}p_{13} = 0$. We have $p_{22} = 0$ and $p_{23} = 0$.

In $p_{32}p_{33} = 0$ we distinguish two cases. First, assume $p_{32} = 0$. Then $p_{12} \neq 0$ (because $|P_{B'B}| \neq 0$). Since $p_{12}p_{13} = 0$ we get $p_{13} = 0$. Use again $|P_{B'B}| \neq 0$ to obtain $p_{33} \neq 0$ and we have proved that, in this case, $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$. Second, assume $p_{32} \neq 0$. Then $p_{33} = 0$ and $p_{13} \neq 0$ because $|P_{B'B}| \neq 0$. Use $p_{12}p_{13} = 0$ to get, reasoning as before, that $p_{12} = 0$ and $p_{32} \neq 0$. This proves again $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.

Now, assume $p_{11} \neq 0$. Then, by (23) and (25), we get $p_{12} = p_{13} = 0$. So, $p_{12}p_{13} = 0$, $p_{32}p_{33} = 0$ and $p_{22}p_{23} = 0$. Now we use (23), (25) and (26) to obtain $p_{31}p_{32} = 0$, $p_{31}p_{33} = 0$ and $p_{32}p_{33} = 0$. Taking into account this identities and $0 \neq |P_{B'B}| = p_{11}p - 22p_{33} - p_{11}p_{32}p_{23}$ we prove $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$ as claimed.

Now that we know the possible matrices for $P_{B'B}$, we may look for all the possible M_B. By Proposition 3.2 (i) all the structure matrices representing the same evolution algebra have the same number of zero entries. This is the reason for studying the classification depending on the number of non-zero entries in M_B (recall that M_B is the matrix given in (19)).

We claim that the first case to be considered is the one for which M_B has four non-zero entries. Indeed, fix our attention in the first and second columns in M_B as given in (19). The maximum number of zero entries in that columns is four. Now, the third column can have only one zero because c_1 and c_2 are non-zero and we have two non-zero entries in the first and second columns, which are neither in the same row nor in the second column as the rank of M_B is two. Taking into account (11), we may assume that two of non-zero entries are 1. In some cases, we will be able to place one or two more 1 in a third and fourth entries. The remaining non-zero entries will be parameters.

A complete description of the procedure can be found in [2]. There we explain the two types of tables we include, called "Table m" and "Table m'". For "Table m", we list in the first row (starting by the second column) the five permutation matrices different from the identity. As for the second row we start with an arbitrary structure matrix under the case we are considering. Then we apply the action of an element in S_3 (listed at the beginning of each column) and write in the corresponding row the obtaining matrix. We start the third row with a matrix under the case we are considering and not included in the second row, and continue in this way until we reach all the possibilities for this case. In order to make easier the understanding of the reader, we distinguish in color (in the web version) the different possibilities that we have. As for the second type of tables we include, the reason is the following: For given parameters (those appearing in the listed matrices), matrices in the same row of a concrete table produce isomorphic evolution algebras. Matrices appearing in different rows correspond to non-isomorphic evolution algebras. Now the question is: For matrices in the same row having different parameters, are the corresponding evolution algebras isomorphic? To answer this question we include in [2] the second type of tables, "Table m'".

Case 1.1 M_B has four non-zero entries.

Note that there is, necessarily, a row with all its entries equal zero, because there is no a column with all its entries equal zero (as $c_1c_2 \neq 0$). For each possible row with three zeros, there are $\binom{6}{2} = 15$ possible places where to put two zeros in the remaining rows. Because A has Property (2LI) a row can not have two zeros. This happens 6 times. We have to eliminate the cases in which there is a zero column (three cases). Then we have $15 - 6 - 3 = 6$ cases for each possible row with three zeros, that is, 18 cases that can be found in [2, Table 2]. Some of them produce isomorphic evolution algebras. Summarizing, there are only four parametric families of mutually non-isomorphic evolution algebras, which are:

$$\left\{\begin{pmatrix}0&0&0\\0&1&1\\1&0&c_1\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&c_2\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&1&0\\1&0&c_1\end{pmatrix}\right\}$$

We study if there are isomorphic evolution algebras in each family. The answer is yes, as shown in [2, Table 2']. The procedure we have used is the following: we start with one M_B and study if there are matrices $P_{B'B}$ such that $M_{B'}$ is in the same family. For the computations we have used Mathematica. The program can be found in an appendix in [2]. This explanation serves for all the cases.

Case 1.2 M_B has five non-zero entries.

The structure matrix must have a zero column. So, for each possible zero row, there exist $\binom{6}{3} = 6$ possible places where to write the remaining zero. Therefore, there are $6 \cdot 3 = 18$ cases that appear in [2, Table 3]. The mutually non-isomorphic parametric families of evolution algebras are three:

$$\left\{\begin{pmatrix}0&0&0\\0&1&1\\1&\alpha&c_1+\alpha\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&c_1+c_2\end{pmatrix},\begin{pmatrix}0&0&0\\1&1&c_1+c_2\\1&0&c_1\end{pmatrix}\right\}$$

For the first family there are cases producing isomorphic evolution algebras. This can be found in [2, Table 3'].

Case 1.3 M_B has six non-zero entries.

There exists $\binom{9}{3} = 84$ possibilities to place three zeros. As the structure matrix has Property (2LI), it can have neither two zeros in a row nor a zero column. So, for each place, we eliminate the six cases where to write two zeros in a row. Then, we remove $9 \cdot 6 + 3 = 57$ cases. Therefore, we have 27 cases. In some of them, the parameters α, β and c_1 must satisfy certain conditions. Concretely, in the fourth row $c_1\alpha\beta + 1 \neq 0$; in the fifth row $c_1\alpha + \beta \neq 0$ and in the seventh row $c_2\alpha + c_1 \neq 0$.

The mutually non-isomorphic parametric families of evolution algebras are seven:

$$\left\{\begin{pmatrix}0&0&0\\1&1&c_1+c_2\\1&\alpha&c_1+c_2\alpha\end{pmatrix},\begin{pmatrix}0&1&1\\0&1&1\\\alpha&0&c_2\alpha\end{pmatrix},\begin{pmatrix}0&\alpha&c_2\alpha\\1&0&1\\1&0&1\end{pmatrix},\begin{pmatrix}0&\alpha&1\\1&0&c_1\\\beta&1&0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0&1&1\\\alpha&\beta&0\\1&0&c_1\end{pmatrix},\begin{pmatrix}1&0&c_1\\0&1&c_2\\0&1&c_2\end{pmatrix},\begin{pmatrix}1&0&c_1\\1&\alpha&0\\0&1&c_2\end{pmatrix}\right\}$$

For the whole procedure see [2, Table 4]. Some of these parametric families of evolution algebras produce isomorphic algebras for different parameters, as can be seen in [2, Table 4'].

Case 1.4 M_B has seven non-zero entries.

There are $\binom{9}{2} = 36$ possibilities to place two zeros. But we have to eliminate the cases in which there are two zeros in the same row. So, there are $36 - 9 = 27$ cases. The parameters c_1, c_2 and β must satisfy certain conditions: in the third and in the fifth rows $c_2\beta + c_1 = 0$. There are six evolution algebras as obtained in [2, Table 5]:

$$\left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}, \begin{pmatrix} 0 & \alpha & c_2\alpha \\ 1 & \beta & 0 \\ 1 & 1 & c_1+c_2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 & c_1\alpha \\ 1 & \beta & 0 \\ 1 & 1 & c_1+c_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ \beta & 0 & \beta \end{pmatrix} \right\}$$

We show in [2, Table 5'], for each parametric family, which one produces isomorphic evolution algebras when we change the parameters.

Case 1.5 M_B has eight non-zero entries.

There are nine possibilities to place a zero in the structure matrix. Therefore, there are two mutually non-isomorphic parametric families of evolution algebras (see [2, Table 6]):

$$\left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ \beta & \gamma & \beta+c_2\gamma \end{pmatrix} \right\}$$

Both parametric families of evolution algebras produce isomorphic algebras under change of parameters; see [2, Table 6'].

Case 1.6 M_B has nine non-zero entries.

The parameters α, β and γ have to verify that the three of them cannot be equal in order for M_B to have rank two. This produces only one parametric family of evolution algebras as shown in [2, Table 7]:

$$\left\{ \begin{pmatrix} 1 & \alpha & c_1+c_2\alpha \\ 1 & \beta & c_1+c_2\beta \\ 1 & \gamma & c_1+c_2\gamma \end{pmatrix} \right\}$$

In [2, Table 7'] we show the change of parameters that produces isomorphic evolution algebras.

Case 2 $c_1 = 0$ and $c_2 \neq 0$.

Note that if we consider the new natural basis $\{e_1', e_2', e_3'\}$ with $e_1' = e_1$, $e_2' = \sqrt{c_2}e_2 - e_3$ and $e_3' = \sqrt{c_2}e_2 + e_3$ we obtain that $(e_2')^2 = (e_3')^2$. By abusing of notation, we may assume that the natural basis concerned is $\{e_1, e_2, e_3\}$ with $e_2^2 = e_3^2$.

Note that in this case the dimension of the annihilator of the evolution algebra is zero. We will see that the possible change of basis matrices are precisely two elements in $S_3 \rtimes (\mathbb{K}^\times)^3$ (we consider only those for which $e_3^2 = e_2^2$) and one more not in $S_3 \rtimes (\mathbb{K}^\times)^3$ that we will specify.

In this case, the equations (23), (24), (25) and (26) are as follows:

$$p_{11}p_{12} = 0; \quad p_{21}p_{22} = -p_{31}p_{32};$$
$$p_{11}p_{13} = 0; \quad p_{21}p_{23} = -p_{31}p_{33};$$
$$p_{12}p_{13} = 0; \quad p_{22}p_{23} = -p_{32}p_{33}.$$

We may suppose that $p_{11} = p_{12} = 0$.

Assume that $p_{31}p_{32}p_{33} \neq 0$. This implies that $p_{21}p_{22}p_{23} \neq 0$. As $p_{21} = \frac{-p_{31}p_{32}}{p_{22}}$, $p_{23} = \frac{p_{33}p_{22}}{p_{32}}$. Then $p_{22} = \pm\sqrt{-1}p_{32}$, $p_{23} = \pm\sqrt{-1}p_{33}$ and $p_{21} = \pm\sqrt{-1}p_{31}$. But, in these conditions $|P_{B'B}| = 0$. Therefore there exists at least one $i \in \{1,2,3\}$ such that $p_{3i} = 0$.

If $p_{31} = 0$, then $p_{21}p_{22} = 0$ and $p_{21}p_{23} = 0$. Since $p_{21} \neq 0$, necessarily $p_{22} = p_{23} = 0$, implying $p_{32}p_{33} = 0$. Consequently, $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.

If $p_{31} \neq 0$ and $p_{32} = 0$, then $p_{21}p_{22} = 0$ and $p_{22}p_{23} = 0$. This implies $p_{21} = p_{23} = p_{33} = 0$ and again $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.

If $p_{31}p_{32} \neq 0$ and $p_{33} = 0$, then $p_{22}p_{23} = 0$ and $p_{21}p_{23} = 0$. Necessarily $p_{23} = 0$. On the other hand, as $p_{31}p_{32} \neq 0$, $p_{21}p_{22} \neq 0$. So, $p_{22} = \frac{-p_{31}p_{32}}{p_{21}}$ and

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & \dfrac{-p_{31}p_{32}}{p_{21}} & 0 \\ p_{31} & p_{32} & 0 \end{pmatrix}, \tag{27}$$

with $p_{13}p_{22}p_{31}p_{32} \neq 0$ and $p_{31}^2 + p_{21}^2 \neq 0$ in order to have $|P_{B'B}| \neq 0$.

If we suppose that $p_{11} = p_{13} = 0$, reasoning in the same way as before, we obtain that the matrices $P_{B'B}$ are in $S_3 \rtimes (\mathbb{K}^\times)^3$ or they are as follows:

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & \dfrac{-p_{33}p_{31}}{p_{21}} \\ p_{31} & 0 & p_{33} \end{pmatrix}, \tag{28}$$

with $p_{12}p_{21}p_{31}p_{33} \neq 0$ and $p_{31}^2 + p_{21}^2 \neq 0$.

Finally, if $p_{12} = p_{13} = 0$, we obtain that the different matrices $P_{B'B}$ that appear are in $S_3 \rtimes (\mathbb{K}^\times)^3$ or are of the form:

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix}, \tag{29}$$

with $p_{11}p_{22}p_{32}p_{33} \neq 0$ and $p_{32}^2 + p_{22}^2 \neq 0$.

If $P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & \dfrac{-p_{31}p_{32}}{p_{21}} & 0 \\ p_{31} & p_{32} & 0 \end{pmatrix}$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21}+\omega_{32}p_{31} & \dfrac{p_{32}^2(\omega_{22}p_{21}+\omega_{32}p_{31})}{p_{21}^2} & \dfrac{p_{13}^2(\omega_{21}p_{21}+\omega_{31}p_{31})}{p_{21}^2+p_{31}^2} \\ \dfrac{p_{21}(\omega_{32}p_{21}-\omega_{22}p_{31})}{p_{32}} & \dfrac{p_{32}(\omega_{32}p_{21}-\omega_{22}p_{31})}{p_{21}} & \dfrac{p_{13}^2p_{21}(\omega_{31}p_{21}-\omega_{21}p_{31})}{p_{32}(p_{21}^2+p_{31}^2)} \\ \dfrac{\omega_{12}(p_{21}^2+p_{31}^2)}{p_{13}} & \dfrac{\omega_{12}p_{32}^2(p_{21}^2+p_{31}^2)}{p_{13}p_{21}^2} & \omega_{11}p_{13} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & \dfrac{-p_{31}p_{33}}{p_{21}} \\ p_{31} & 0 & p_{33} \end{pmatrix}$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21}+\omega_{32}p_{31} & \dfrac{p_{12}^2(\omega_{21}p_{21}+\omega_{31}p_{31})}{p_{21}^2+p_{31}^2} & \dfrac{p_{33}^2(\omega_{22}p_{21}+\omega_{32}p_{31})}{p_{21}^2} \\ \dfrac{\omega_{12}(p_{21}^2+p_{31}^2)}{p_{12}} & \omega_{11}p_{12} & \dfrac{\omega_{12}p_{33}^2(p_{21}^2+p_{31}^2)}{p_{12}p_{21}^2} \\ \dfrac{p_{21}(\omega_{32}p_{21}-\omega_{22}p_{31})}{p_{33}} & \dfrac{p_{12}^2p_{21}(\omega_{31}p_{21}-\omega_{21}p_{31})}{p_{33}(p_{21}^2+p_{31}^2)} & \dfrac{p_{33}(\omega_{32}p_{21}-\omega_{22}p_{31})}{p_{21}} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix}$ then

$$M_{B'} = \begin{pmatrix} \omega_{11}p_{11} & \dfrac{\omega_{12}(p_{22}^2+p_{32}^2)}{p_{11}} & \dfrac{\omega_{12}p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \dfrac{p_{11}^2(\omega_{21}p_{22}+\omega_{31}p_{32})}{p_{22}^2+p_{32}^2} & \omega_{22}p_{22}+\omega_{32}p_{32} & \dfrac{p_{33}^2(\omega_{22}p_{22}+\omega_{32}p_{32})}{p_{22}^2} \\ \dfrac{p_{11}^2p_{22}(\omega_{31}p_{22}-\omega_{21}p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \dfrac{p_{22}(\omega_{32}p_{22}-\omega_{22}p_{32})}{p_{33}} & \dfrac{p_{33}(\omega_{32}p_{22}-\omega_{22}p_{32})}{p_{22}} \end{pmatrix}.$$

Taking into account that we were assuming $e_3^2 = e_2^2$, then the possible change of basis matrices are the following:

$$\left\{ \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \\ 0 & p_{32} & 0 \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix} \mid p_{11}, p_{22}, p_{32}, p_{33} \in \mathbb{K}^{\times} \right\}. \tag{30}$$

In what follows we will classify in three steps: we start by taking into account the first two families of change of basis matrices of the set (30) which leave invariant the number of non-zero entries in the first and second columns. Then, we will analyze if the resulting families of evolution algebras are or not isomorphic under the action of one matrix of the third family in (30), i.e., we will see if some families of evolution algebras are included into other families when applying the change of basis matrices of the third type. Finally, we will analyze, for each of the resulting parametric families, if their algebras are mutually isomorphic.

We list the different matrices into tables taking in account the number of zeros in the first and second columns. Each of these tables will receive the name of "Figure m". According to (11) we will write as many 1 as possible and the others non-zero entries will be arbitrary parameters α, β, γ and λ under the restriction $e_2^2 = e_3^2$. We start by the first one and applying the action of the elements:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix}$$

with $p_{11}, p_{22}, p_{32}, p_{33} \in \mathbb{K}^\times$ and $p_{32}^2 + p_{22}^2 \neq 0$.

Case 2.1 M_B has two non-zero entries in the first and second columns.

There are $\binom{6}{4} = 15$ possible places where to put four zeros. Since some of the resulting matrices have rank 1, they must be removed from the 15 cases. This happens whenever the first or the second columns is zero (2 cases) and the remaining zeros can be settled in three different places. This produces 6 cases. We also eliminate the cases in which two different rows are zero (3 options). Therefore we have $15 - 6 - 3 = 6$ different matrices classified in 3 types. Their structure matrices appear in the first column of the table called Figure 1 in [2].

Case 2.2 M_B has three non-zero entries in the first and second columns.

There exist $\binom{6}{3} = 20$ possible places where to write three zeros. We remove the matrices which have rank 1. This happens 2 times: when the first or the second column is zero. Therefore we have $20 - 2 = 18$ cases. There are 10 types listed in the tables called Figure 2 in [2].

Case 2.3 M_B has four non-zero entries in the first and second columns.

There exists $\binom{6}{3} = 15$ possible places where to write four zeros. The non-zero parameters α, β satisfy that $\alpha \neq \beta$ in the matrices appearing as types 14 and 20. This is because the rank of those matrices has to be two. There are nine different types. They are listed in the tables called Figure 3 in [2].

Case 2.4 M_B has five non-zero entries in the first and second columns.

There are only 6 possibilities: those for which we place only one zero in one place of the first column or of the second column. There are four types which are listed in the table called Figure 4 in [2].

Case 2.5 M_B has six non-zero entries in the first and second columns.

The condition that the entries of the matrix must satisfy is one of the following: $\alpha \neq \beta$, or $\lambda \neq \gamma$ or $\alpha\lambda \neq \beta\gamma$. There is only one possibility listed in the table called Figure 5 in [2].

These tables give us a first classification, that can be redundant in some cases. Now we study if algebras having different types are isomorphic or not. The last step will be to study if algebras in the same type are isomorphic.

- The evolution algebra given in Type 1 is included in the parametric family of algebras of Type 4.
- The evolution algebra given in Type 2 is included in the parametric family of algebras of Type 11.
- The evolution algebra given in Types 3, 12 and 13 are included in the parametric family of algebras of Type 20.
- The parametric families of evolution algebras given in Types 5, 10, 16 and 17 are included in the parametric family of algebras of Type 23.
- The parametric families of evolution algebras given in Types 6, 7, 19 and 22 are included in the parametric family of algebras of Type 25.
- The parametric family of evolution algebras given in Type 8 is included in the parametric family of algebras of Type 21.
- The parametric family of evolution algebras given in Type 9 is included in the parametric family of algebras of Type 18.
- The parametric families of evolution algebras given in Types 14, 15, 24 and 26 are included in the parametric family of algebras of Type 27.

Therefore, there are eight subtypes of parametric families of evolution algebras, which are listed below.

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \lambda & \lambda \end{pmatrix} \right\}$$

Remark 3.7. Note that these matrices are precisely those appearing in the tables called Figures for which the change of basis matrices of type Q leaves invariant the number of non-zero entries and its place in the structure matrix. This does not mean that the number of non-zero entries is preserved (see, for example, in Figure 2, that the first matrix of Type 5 has four non-zero entries while the third matrix in the same line has seven).

Now, we will analyze when the resulting parametric families of evolution algebras are mutually isomorphic. In some cases, we will reduce the number of parameters and some of these parametric families will be isomorphic to one of the known evolution algebras.

Every evolution algebra with structure matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha \end{pmatrix}$ satisfying $\alpha^2+1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Indeed, if $\alpha \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1+\alpha}{1+\alpha^2} & \dfrac{1-\alpha}{1+\alpha^2} \\ 0 & \dfrac{-1+\alpha}{1+\alpha^2} & \dfrac{1+\alpha}{1+\alpha^2} \end{pmatrix}.$$

In case of $\alpha = -1$, we assume the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{31}$$

The evolution algebra with structure matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$ satisfying $\alpha^2+1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Indeed, if $\alpha \neq 1, -1$, we take the change of basis matrix

$$\begin{pmatrix} \sqrt[3]{\frac{2}{1+\alpha^2}} & 0 & 0 \\ 0 & \dfrac{1-\alpha}{\sqrt[3]{2(1+\alpha^2)^2}} & \dfrac{1+\alpha}{\sqrt[3]{2(1+\alpha^2)^2}} \\ 0 & \dfrac{1+\alpha}{\sqrt[3]{2(1+\alpha^2)^2}} & \dfrac{\alpha-1}{\sqrt[3]{2(1+\alpha^2)^2}} \end{pmatrix}.$$

If $\alpha = -1$, we consider again the change of basis matrix given in (31).

Every evolution algebra with structure matrix $\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}$ satisfying $\beta^2+1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} \alpha' & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ for some $\alpha' \in \mathbb{K}$. Indeed, if $\beta \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} \dfrac{2}{1+\beta^2} & 0 & 0 \\ 0 & \dfrac{1-\beta}{1+\beta^2} & \dfrac{1+\beta}{1+\beta^2} \\ 0 & \dfrac{1+\beta}{1+\beta^2} & \dfrac{\beta-1}{1+\beta^2} \end{pmatrix}.$$

In case of $\beta = -1$, we can also consider the change of basis matrix given in (31).

The evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}$ satisfying $\beta^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha' & 1 & 1 \end{pmatrix}$ for some $\alpha' \in \mathbb{K}$. Indeed, if $\beta \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} \frac{\sqrt{2}}{\sqrt{1+\alpha+\beta(-1+\alpha)}} & 0 & 0 \\ 0 & \frac{1-\beta}{1+\beta^2} & \frac{1+\beta}{1+\beta^2} \\ 0 & \frac{1+\beta}{1+\beta^2} & \frac{\beta-1}{1+\beta^2} \end{pmatrix}.$$

In case of $\beta = -1$, we take again the change of basis matrix given in (31).

Every evolution algebra with structure matrix $\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}$ satisfying $\alpha^2 + \beta^2 \neq 0$ and $\beta^2 \neq 1$ is isomorphic to the evolution algebra having structure matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \beta' & 0 & 0 \end{pmatrix}$, for some $\beta' \in \mathbb{K}$. Indeed, if $\alpha \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha-\beta s}{\alpha^2+\beta^2} & \frac{-(\beta+\alpha s)}{\alpha^2+\beta^2} \\ 0 & \frac{\beta+\alpha s}{\alpha^2+\beta^2} & \frac{\alpha-\beta s}{\alpha^2+\beta^2} \end{pmatrix},$$

where $s = \sqrt{-1+\alpha^2+\beta^2}$.

For $\alpha = -1$, consider the change of basis matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

On the other hand, every evolution algebra with structure matrix $\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (β=1) and $\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ ($\beta = -1$) is isomorphic to the evolution algebra with structure matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}$. Indeed, take the new natural bases $\{e_1, e_3, e_2\}$ and $\{e_1, -e_3, e_2\}$, respectively.

The evolution algebra with structure matrix $\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}$ with $\gamma^2+1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} 1 & 0 & 0 \\ \alpha' & 1 & 1 \\ \beta' & 1 & 1 \end{pmatrix}$ for certain $\alpha', \beta' \in \mathbb{K}$. Indeed, if $\gamma \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1+\gamma}{1+\gamma^2} & \dfrac{1-\gamma}{1+\gamma^2} \\ 0 & \dfrac{\gamma-1}{1+\gamma^2} & \dfrac{1+\gamma}{1+\gamma^2} \end{pmatrix}.$$

If $\gamma = -1$, take again the change of basis matrix (31).

The evolution algebra with structure matrix $\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}$ with $\gamma^2+1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} 0 & 1 & 1 \\ \alpha' & 1 & 1 \\ \beta' & 1 & 1 \end{pmatrix}$ for certain $\alpha', \beta' \in \mathbb{K}$. Indeed, if $\gamma \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} \dfrac{2}{1+\gamma^2} & 0 & 0 \\ 0 & \dfrac{1+\gamma}{1+\gamma^2} & \dfrac{1-\gamma}{1+\gamma^2} \\ 0 & \dfrac{\gamma-1}{1+\gamma^2} & \dfrac{1+\gamma}{1+\gamma^2} \end{pmatrix}.$$

If $\gamma = -1$, also we take the change of basis matrix given in (31).

The evolution algebra with structure matrix $\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \lambda & \lambda \end{pmatrix}$ with $\lambda^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix $\begin{pmatrix} \alpha' & 1 & 1 \\ \beta' & 1 & 1 \\ \gamma' & 1 & 1 \end{pmatrix}$ for certain $\alpha', \beta', \gamma' \in \mathbb{K}$. Indeed, if $\lambda \neq -1$, we take the change of basis matrix:

$$\begin{pmatrix} \dfrac{2}{1+\lambda^2} & 0 & 0 \\ 0 & \dfrac{1+\lambda}{1+\lambda^2} & \dfrac{1-\lambda}{1+\lambda^2} \\ 0 & \dfrac{\lambda-1}{1+\lambda^2} & \dfrac{1+\lambda}{1+\lambda^2} \end{pmatrix}.$$

If $\gamma = -1$, we consider again the change of basis matrix determined in (31).

Summarizing, whenever $e_3^2 = e_2^2$ we obtain the following families of evolution algebras which are classified depending on the non-zero entries of the matrices in S (see Tables 8–13).

Table 8
$\dim(A^2) = 2$; $\dim(\mathrm{ann}(A)) = 0$; A has not Property (2LI); four non-zero entries of the matrices in S.

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \sqrt{-1} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}$

Table 9
$\dim(A^2) = 2$; $\alpha \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has not Property (2LI); five non-zero entries of the matrices in S.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ \alpha\sqrt{-1} & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ -\alpha\sqrt{-1} & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$

Table 10
$\dim(A^2) = 2$; $\alpha \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has not Property (2LI); six non-zero entries of the matrices in S.

$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

Table 11
$\dim(A^2) = 2$; $\alpha\beta \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has not Property (2LI); seven non-zero entries of the matrices in S.

$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

Table 12
$\dim(A^2) = 2$; $\alpha\beta \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has not Property (2LI); eight non-zero entries of the matrices in S.

$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

We have included the study of the isomorphisms under change of parameters in the Tables 9'–13' in [2].

Case 3 Assume $c_2 = 0, c_1 \neq 0$.

Considering the natural basis $B' = \{e_2, e_1, e_3\}$ we obtain the following structure matrix:

$$M_{B'} = \begin{pmatrix} \omega_{22} & \omega_{21} & c_1\omega_{21} \\ \omega_{12} & \omega_{11} & c_1\omega_{11} \\ \omega_{32} & \omega_{31} & c_1\omega_{31} \end{pmatrix},$$

Table 13
$\dim(A^2) = 2$; $\alpha\beta\gamma \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has not Property (2LI); nine non-zero entries of the matrices in S.

$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

and now we are in the same conditions as in Case 2.

Case 4 Suppose $c_1 = c_2 = 0$.

Recall by (19) that the structure matrix is

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}.$$

Remark 3.8. In what follows we are going to prove that the number of zero entries in the first and in the second rows in the structure matrix is preserved by any change of basis.

With the explained goal in mind, we study all the possible change of basis matrices. Let B' be another natural basis and consider the change of basis matrix $P_{B'B}$. The equations (23), (24), (25) and (26) give:

$$p_{11}p_{12} = 0; \quad p_{21}p_{22} = 0;$$
$$p_{11}p_{13} = 0; \quad p_{21}p_{23} = 0;$$
$$p_{12}p_{13} = 0; \quad p_{22}p_{23} = 0.$$

It is easy to check that $P_{B'B}$ has two zero entries in the first and the second rows. Moreover, since $|P_{B'B}| \neq 0$, necessarily $p_{1i}p_{2j} \neq 0$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. We distinguish the six different cases that appear in order to study the structure matrix $M_{B'}$.

If $P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ with $p_{11}p_{22}p_{33} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{11}p_{11} & \dfrac{\omega_{12}p_{22}^2}{p_{11}} & 0 \\ \dfrac{\omega_{21}p_{11}^2}{p_{22}} & \omega_{22}p_{22} & 0 \\ \dfrac{p_{11}(\omega_{31}p_{11}p_{22} - \omega_{11}p_{31}p_{22} - \omega_{21}p_{11}p_{32})}{p_{22}p_{33}} & \dfrac{p_{22}(\omega_{32}p_{11}p_{22} - \omega_{12}p_{31}p_{22} - \omega_{22}p_{11}p_{32})}{p_{11}p_{33}} & 0 \end{pmatrix}. \tag{32}$$

If $P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ for $p_{11}p_{23}p_{32} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{11}p_{11} & 0 & \frac{\omega_{12}p_{23}^2}{p_{11}} \\ \frac{p_{11}(\omega_{31}p_{11}p_{23} - \omega_{11}p_{31}p_{23} - \omega_{21}p_{11}p_{33})}{p_{23}p_{32}} & 0 & \frac{p_{23}(\omega_{32}p_{11}p_{23} - \omega_{12}p_{31}p_{23} - \omega_{22}p_{11}p_{33})}{p_{11}p_{32}} \\ \frac{\omega_{21}p_{11}^2}{p_{23}} & 0 & \omega_{22}p_{23} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ where $p_{12}p_{21}p_{33} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21} & \frac{\omega_{21}p_{12}^2}{p_{21}} & 0 \\ \frac{\omega_{12}p_{21}^2}{p_{12}} & \omega_{11}p_{12} & 0 \\ \frac{p_{21}(\omega_{32}p_{12}p_{21} - \omega_{12}p_{32}p_{21} - \omega_{22}p_{12}p_{31})}{p_{12}p_{33}} & \frac{p_{12}(\omega_{31}p_{12}p_{21} - \omega_{11}p_{32}p_{21} - \omega_{21}p_{12}p_{31})}{p_{21}p_{33}} & 0 \end{pmatrix}. \tag{33}$$

If $P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ with $p_{12}p_{23}p_{31} \neq 0$ then

$$M_{B'} = \begin{pmatrix} 0 & \frac{p_{12}(\omega_{31}p_{12}p_{23} - \omega_{11}p_{32}p_{23} - \omega_{21}p_{12}p_{33})}{p_{23}p_{31}} & \frac{p_{23}(\omega_{32}p_{12}p_{23} - \omega_{12}p_{32}p_{23} - \omega_{22}p_{12}p_{33})}{p_{12}p_{31}} \\ 0 & \omega_{11}p_{12} & \frac{\omega_{12}p_{23}^2}{p_{12}} \\ 0 & \frac{\omega_{21}p_{12}^2}{p_{23}} & \omega_{22}p_{23} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ for $p_{13}p_{21}p_{32} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21} & 0 & \frac{\omega_{21}p_{13}^2}{p_{21}} \\ \frac{p_{21}(\omega_{32}p_{13}p_{21} - \omega_{12}p_{33}p_{21} - \omega_{22}p_{13}p_{31})}{p_{13}p_{32}} & 0 & \frac{p_{13}(\omega_{31}p_{13}p_{21} - \omega_{11}p_{33}p_{21} - \omega_{21}p_{13}p_{31})}{p_{21}p_{32}} \\ \frac{\omega_{12}p_{21}^2}{p_{13}} & 0 & \omega_{11}p_{13} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ where $p_{13}p_{21}p_{32} \neq 0$ then

$$M_{B'} = \begin{pmatrix} 0 & \frac{p_{22}(\omega_{32}p_{13}p_{22} - \omega_{12}p_{33}p_{22} - \omega_{22}p_{13}p_{32})}{p_{13}p_{31}} & \frac{p_{13}(\omega_{31}p_{13}p_{22} - \omega_{11}p_{33}p_{22} - \omega_{21}p_{13}p_{32})}{p_{22}p_{31}} \\ 0 & \omega_{22}p_{22} & \frac{\omega_{21}p_{13}^2}{p_{22}} \\ 0 & \frac{\omega_{12}p_{22}^2}{p_{13}} & \omega_{11}p_{13} \end{pmatrix}.$$

Note that we only have to take in to account the change of basis matrices which transform a structure matrix having the third column equals zero into another one of the same type. These are those $P_{B'B}$ appearing in the first and in the third cases. We denote them by Q' and by Q'', respectively. Looking at the different $M_{B'}$ that appear, we obtain the claim.

Then, if we omit the structure matrices which can be obtained from the permutation $(1,2)$, the only possibilities are:

$$\left\{\begin{pmatrix}\omega_{11}&0&0\\0&\omega_{22}&0\\\omega_{31}&\omega_{32}&0\end{pmatrix},\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix},\begin{pmatrix}\omega_{11}&0&0\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&0\end{pmatrix},\begin{pmatrix}0&\omega_{12}&0\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&0\end{pmatrix},\begin{pmatrix}\omega_{11}&\omega_{12}&0\\\omega_{21}&\omega_{22}&0\\\omega_{31}&\omega_{32}&0\end{pmatrix}\right\}\bigcup$$
$$\left\{\begin{pmatrix}\omega_{11}&0&0\\0&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix},\begin{pmatrix}\omega_{11}&\omega_{12}&0\\0&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix},\begin{pmatrix}\omega_{11}&0&0\\\omega_{21}&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix}\begin{pmatrix}0&\omega_{12}&0\\0&0&0\\\omega_{31}&\omega_{32}&0\end{pmatrix}\right\}$$

with $\omega_{ij}\neq 0$ for $i,j\in\{1,2\}$.

According to (32) and (33), we claim that we can remove the third row of the structure matrices of the first set and write 0 if and only if $\omega_{11}\omega_{22}-\omega_{12}\omega_{21}\neq 0$. For the matrix (32) we consider $p_{11}=p_{22}=1$ and we have

$$\omega_{31}p_{11}p_{22}-\omega_{11}p_{31}p_{22}-\omega_{21}p_{11}p_{32}=\omega_{31}-\omega_{11}p_{31}-\omega_{21}p_{32}=0;$$
$$\omega_{32}p_{11}p_{22}-\omega_{12}p_{31}p_{22}-\omega_{22}p_{11}p_{32}=\omega_{31}-\omega_{12}p_{31}-\omega_{22}p_{32}=0.$$

So, this linear system has solution if $\omega_{11}\omega_{22}-\omega_{12}\omega_{21}\neq 0$.

If we take (33), we reason in the same way and our claim has been proved.

Now we can place 0 instead of ω_{31} in the first three matrices of the second set. Indeed, as in these structure matrices $\omega_{11}\neq 0$ and supposing $p_{11}=p_{22}=1$ we have the equation $\omega_{31}-\omega_{11}p_{31}-\omega_{21}p_{32}=0$ if $\omega_{21}\neq 0$ and $\omega_{31}-\omega_{11}p_{31}=0$ if $\omega_{21}=0$. In any case, the equations have always solution.

In the last structure matrix of the second set we can write 0 instead of ω_{32}. For this, it is enough to take $p_{22}=p_{11}=0$ and $p_{31}=\frac{\omega_{32}}{\omega_{12}}$.

Finally, we can obtain the maximum number of entries equal 1 by using (11). When placing 1 is not possible we write the parameters α, β and γ. Summarizing, there are ten possibilities which are:

$$\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&0\\0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&0&0\\0&1&0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}\alpha&0&0\\1&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&\alpha&0\\1&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&0\\\alpha&\beta&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&0\\\alpha&\alpha&0\\1&\beta&0\end{pmatrix}\right\}.$$

We develop the whole procedure in [2, Tables 14–17]. We study in [2, Tables 16' and 17'] if each resulting family contains isomorphic evolution algebras. We remark that in "Table m'" the elements p_{ij} have to satisfy the necessary conditions in order for $P_{B'B}$ to have rank 3.

Case $\dim(A^2)=3$**.**

In order to classify all the possible matrices corresponding to structure matrices of three-dimensional evolution algebras A such that $A^2 = A$ (equivalently $\dim(A^2) = 3$), we will use Proposition 3.2. Notice that in this case the number of zeros in all the structure matrices of a given evolution algebra is invariant (see Proposition 3.2 (i)). Equivalently, the number of non-zero entries is invariant. This is the reason because of which we will classify taking into account this last number. Note that the minimum number of non-zero entries in M_B is exactly three.

Case 1. M_B has three non-zero elements.

We compute the determinant of M_B.

$$|M_B| = \omega_{11}\omega_{22}\omega_{33} + \omega_{12}\omega_{23}\omega_{31} + \omega_{13}\omega_{21}\omega_{32} - \omega_{13}\omega_{22}\omega_{31} - \omega_{21}\omega_{12}\omega_{33} - \omega_{11}\omega_{32}\omega_{23}. \quad (34)$$

Since $|M_B| \neq 0$, only one of the six summands is non-zero. Assume, for example, $\omega_{12}\omega_{23}\omega_{31} \neq 0$. Take $\alpha = \frac{1}{\sqrt[7]{\omega_{12}\omega_{23}^2\omega_{31}^4}}$, $\beta = \alpha^4\omega_{23}\omega_{31}^2$ and $\gamma = \alpha^2\omega_{31}$. Then

$$(\alpha, \beta, \gamma) \cdot \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Reasoning in this way with $\omega_{1\sigma(1)}\omega_{2\sigma(2)}\omega_{3\sigma(3)}$ (where $\sigma \in S_3$) instead of with $\omega_{12}\omega_{23}\omega_{31}$, we obtain a natural basis B' such that $M_{B'} = (\varpi_{ij})$, with $\varpi_{i\sigma(i)} = 1$ and $\varpi_{ij} = 0$ for any $j \neq \sigma(i)$.

This justifies that these are the only matrices we consider in order to get the classification. Notice that there are only six. Since we do not know which of them are in the same orbit (considering the action described in Section 3.1), we start with one of them, say M, and consider $\tau \cdot M$ for any $\tau \in S_3$.

We display $\{\tau \cdot M \mid \tau \in S_3\}$ in a row. Then, we start with another of these matrices, say M', not appearing in this row, and display $\{\tau \cdot M' \mid \tau \in S_3\}$ in a second row. We continue in this way until we get the six different matrices. We color these six matrices to make easier the reader to find them (see Table 18).[8]

Therefore, there are only three orbits and, consequently, only three evolution algebras A in the case we are studying. Their structure matrices are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (35)$$

Case 2. M_B has four non-zero elements.

Reasoning as in Case 1, we arrive at a natural basis B' of the evolution algebra A such that $M_{B'} = (\varpi_{ij})$, with $\varpi_{i\sigma(i)} = 1$, $\varpi_{ij} \neq 0$ for some $j \neq \sigma(i)$ and $\varpi_{ik} = 0$ for every $k \neq \sigma(i), j$ for every permutation $\sigma \in S_3$.

[8] The color version of Table 18 will appear on the web.

Table 18
$\dim(A^2) = 3$; three non-zero entries.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

In order to describe the matrices producing non-isomorphic evolution algebras, first, we notice the following. Given a matrix as explained below, no matter where we put the four non-zero elements (three 1 and one arbitrary parameter μ which has to be non-zero) that the resulting matrices correspond to isomorphic evolution algebras. This is because we will not be worried about where to place the parameter. Then we explain which are the possible cases.

We have to put five 0 into nine places (the nine entries of the matrix). This can be done in $\binom{9}{5} = 126$ ways. But we must remove the cases in which $|M_{B'}| = 0$. This happens:

(a) When the entries of a row are zero.
(b) When the entries of a column are zero but there is no a row which consists of zeros.
(c) When the matrix has a 2×2 minor with every entry equals zero and it has not a row or a column of zeros.

These three cases are mutually exclusive.

(a) The cases in which there is a column of zeros are $3\binom{6}{2} = 45$ (3 corresponds to the three columns and $\binom{6}{2}$ corresponds to the different ways in which two zeros can be distributed in the six remaining places).

(b) For the rows the reasoning in similar: we have 45 cases. Now we have to take into account that there are cases which have been considered twice (just when there is a row and a column which are zero). This happens 9 times. Therefore, we have $45 - 9 = 36$ options in this case.

(c) Once the matrix has a 2×2 minor with every entry equals zero, the fifth zero must be only in one place if we want to avoid the matrix having a row or column of zeros. There are 9 options to put a zero in a matrix. Once this happens, we remove the corresponding row and the corresponding column and there are four places where to put four zeros. Hence, there are 9 possibilities in this case.

Taking into account (a), (b) and (c), there are $126 - (45 + 36 + 9) = 36$ different matrices we can consider.

As in Case 1, we list all the options in a table. The elements that appear in every row correspond to the action of every element of S_3 on the matrix placed first. There exist six mutually non-isomorphic parametric families of evolution algebras, which are:

$$\left\{\begin{pmatrix}1&\mu&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}\mu&1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\\mu&0&1\end{pmatrix},\begin{pmatrix}\mu&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&\mu&1\\1&0&0\\0&1&0\end{pmatrix}\right\}.$$

The study of isomorphisms between elements of the same parametric family of evolution algebras can be found in [2, Table 19'].

Case 3. M_B has five non-zero elements.

We proceed as in the cases above and obtain that in order to classify we need to consider only matrices with four zero entries and five non-zero entries. By changing the basis, we may assume that three of the elements are 1 and the other are arbitrary parameters λ and μ, with the only restriction of being non-zero and such that $\lambda\mu \neq 1$ (this condition is needed because the determinant must be non-zero).

The different matrices to be considered are those for which we place four zeros: $\binom{9}{4} = 126$. On the one hand, we must remove those for which there is a row or a column which are zero (because these matrices have zero determinant). If one row or column consists of zeros, then the fourth zero can be placed in six different positions. Since there are 3 rows and 3 columns, this happens 6 times. On the other hand, we must remove those for which there is a 2×2 minor with every entry equals zero. Consequently, we have $126 - 6^2 - 9 = 81$ cases that we display in the table that follows. The number of mutually non-isomorphic parametric families of evolution algebras is sixteen:

$$\left\{\begin{pmatrix}1&\mu&\lambda\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&\mu&0\\\lambda&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&\mu&0\\0&1&\lambda\\0&0&1\end{pmatrix},\begin{pmatrix}1&\mu&0\\0&1&0\\0&\lambda&1\end{pmatrix},\begin{pmatrix}\mu&1&\lambda\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}\mu&1&0\\1&0&\lambda\\0&0&1\end{pmatrix},\right.$$
$$\begin{pmatrix}\mu&1&0\\1&0&0\\\lambda&0&1\end{pmatrix},\begin{pmatrix}\mu&1&0\\1&0&0\\0&\lambda&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\1&0&\lambda\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&\mu\\\lambda&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&\mu\\0&\lambda&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\\mu&\lambda&1\end{pmatrix},$$
$$\left.\begin{pmatrix}\mu&\lambda&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}\mu&0&1\\1&\lambda&0\\0&1&0\end{pmatrix},\begin{pmatrix}\mu&0&1\\1&0&0\\\lambda&1&0\end{pmatrix},\begin{pmatrix}0&\mu&1\\1&0&\lambda\\0&1&0\end{pmatrix}\right\}.$$

We show the study of isomorphism of parametric families of evolution algebras when we change the parameters in Table 20' in [2].

Case 4. M_B has six non-zero elements.

Once again we reason in the same way and we can fix our attention in those matrices with three zeros and six non-zero entries.

The different possibilities are: $\binom{9}{3} - 6 = 78$. Note that $\binom{9}{3}$ are the different ways of placing 3 zeros in a 3×3 matrix while 6 corresponds to the cases in which there is a row or a column which is zero.

Making changes on the elements of the basis we may consider three entries equals 1. The only restrictions on the other three elements, say λ, μ and ρ, which must be non-zero, are the needed ones in order to not have zero determinant. This means $\mu\rho \neq 1$, $\lambda\rho \neq 1$, $\mu\lambda \neq 1$ and $\mu\rho\lambda \neq -1$.

There are fifteen parametric families of evolution algebras, which are:

$$\left\{\begin{pmatrix}1&\mu&\lambda\\ \rho&1&0\\ 0&0&1\end{pmatrix},\begin{pmatrix}1&\mu&\lambda\\ 0&1&\rho\\ 0&0&1\end{pmatrix},\begin{pmatrix}1&\mu&0\\ 0&1&\lambda\\ \rho&0&1\end{pmatrix},\begin{pmatrix}1&\mu&0\\ \lambda&1&0\\ \rho&0&1\end{pmatrix},\begin{pmatrix}\mu&1&\lambda\\ 1&0&\rho\\ 0&0&1\end{pmatrix},\begin{pmatrix}\mu&1&\lambda\\ 1&0&0\\ \rho&0&1\end{pmatrix},\right.$$
$$\begin{pmatrix}\mu&1&\lambda\\ 1&0&0\\ 0&\rho&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\ 1&\lambda&0\\ \rho&0&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\ 1&\lambda&0\\ 0&\rho&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\ 1&0&\lambda\\ \rho&0&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\ 1&0&0\\ \lambda&\rho&1\end{pmatrix},\begin{pmatrix}0&1&0\\ 1&\mu&0\\ \lambda&\rho&1\end{pmatrix},$$
$$\left.\begin{pmatrix}\mu&\lambda&1\\ 1&\rho&0\\ 0&1&0\end{pmatrix},\begin{pmatrix}\mu&\lambda&1\\ 1&0&0\\ \rho&1&0\end{pmatrix},\begin{pmatrix}0&\mu&1\\ 1&0&\lambda\\ \rho&1&0\end{pmatrix}\right\}.$$

We show in [2, Table 21'] which of the parametric families of evolution algebras produce isomorphic algebras when changing the parameters.

Case 5. M_B has seven non-zero elements.

The different cases that we must consider are $\binom{9}{2} = 36$. Every matrix has three entries which are 1 and four non-zero parameters $\delta, \lambda, \mu, \rho$, which must satisfy one of the following conditions, depending on the case we are considering, in order for the matrix to not have zero determinant: $\mu\rho \neq 1$; $\mu\rho + \delta\lambda \neq 1$; $\delta\mu \neq 1$; $\delta\mu + \lambda\rho \neq 1$; $\delta\lambda \neq 1$; $\delta\rho - \delta\lambda\mu \neq 1$; $\delta\rho \neq 1$; $\mu\rho - \delta\lambda\rho \neq 1$.

The number of mutually non-isomorphic parametric families of evolution algebras is eight, which are:

$$\left\{\begin{pmatrix}1&\mu&\lambda\\ \rho&1&\delta\\ 0&0&1\end{pmatrix},\begin{pmatrix}1&\mu&\lambda\\ \rho&1&0\\ \delta&0&1\end{pmatrix},\begin{pmatrix}1&\mu&\lambda\\ \rho&1&0\\ 0&\delta&1\end{pmatrix},\begin{pmatrix}1&0&\mu\\ \lambda&1&\rho\\ \delta&0&1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}\mu&1&\lambda\\ 1&0&0\\ \rho&\delta&1\end{pmatrix},\begin{pmatrix}\mu&1&\lambda\\ 1&0&\rho\\ \delta&0&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\ 1&\lambda&\rho\\ \delta&0&1\end{pmatrix},\begin{pmatrix}0&1&\mu\\ 1&0&\lambda\\ \rho&\delta&1\end{pmatrix}\right\}.$$

We show in [2, Table 22'] which of the parametric families give isomorphic evolution algebras under change of parameters.

Case 6. There are eight non-zero elements in the matrix.

In this case there are only nine possibilities which appear in the table that follows. The condition that the entries of the matrix must satisfy is one of the following: $\eta\lambda + \mu\rho - \delta\eta\mu \neq 1$ or $\delta\mu + \eta\rho - \delta\eta\lambda \neq 1$, just to be sure that the determinant of the corresponding matrix is different from zero. There are two parametric families of evolution algebra, which are:

$$\left\{\begin{pmatrix}1&\mu&\lambda\\ \rho&1&\delta\\ \eta&0&1\end{pmatrix},\begin{pmatrix}\mu&\lambda&1\\ \rho&1&\delta\\ 1&\eta&0\end{pmatrix}\right\}.$$

We can see when the parametric evolution algebras are isomorphic when we change the parameters in Table 23' in [2].

Case 7 All the entries in the matrix are non-zero. In this case only one matrix appears:

$$\left\{ \begin{pmatrix} 1 & \mu & \lambda \\ \rho & 1 & \delta \\ \eta & \tau & 1 \end{pmatrix} \right\}$$

and the condition that the parameters must satisfy is $\eta\rho + \delta\lambda + \mu\tau - \eta\lambda\tau - \delta\mu\rho \neq 1$.

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