On Degenerations of Lie Superalgebras

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Universidade Federal do ABC, Santo André, S.P., Brasil, 2018


Mérida,
Yucatán
México,
D.F

Plan of the Talk

Convention

Throughout this work all vector spaces are finite dimensional over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Algebras

Definition

An $\mathbb{F}$-algebra is a vector space A with a bilinear map

$$\cdot : A \times A \to A.$$

We say that A has a unit element if there is an element $1 \in A$ such that

$$1 \cdot a = a \cdot 1 = a, \quad \text{for all } a \in A.$$

Example

- $\mathbb{F}[x]$ is a commutative, associative algebra with unit.
- Let V a vector space.

$$\mathrm{End}(V) = \{T : V \to V \mid T \text{ is linear}\}$$

is an associative algebra with unit (non-commitative).

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- $C^n(\mathbb{R}) : \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is n-times differentiable}\}, n \in \mathbb{N} \cup \{0\}$.

- $\mathbb{R}^3$ with the cross product is a non-commutative, non-associative algebra without unit. In fact:

$$(e_1 \times e_2) \times e_2 \neq e_1 \times (e_2 \times e_2).$$

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Algebras defined by identities

Observation. An Algebra A is

- commutative if $xy = yx$, for all $x, y \in A$, i.e.,

$$P_1(x, y) = xy - yx = 0, \ \forall\, x, y \in A.$$

- associative if

$$P_2(x, y, z) = (xy)z - x(yz) = 0, \ \forall\, x, y, z \in A.$$

Notation. Let V be an n-dimensional $\mathbb{F}$-vector space, and $P_1, \ldots, P_n$ polynomial identities.

$$\mathcal{A}lg_n(P_1, \ldots, P_r)$$

is the variety of algebras defined by $P_1, \ldots, P_n$.

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An algebra $\mathfrak{g}$, with product $[\cdot,\ \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, is called a Lie algebra if:

(i) $[x, y] = -[y, x]$ (antisymmetry),

(ii) $[[x, y,], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity)

for all $x, y, z \in \mathfrak{g}$.

Remark. (i) is equivalent to the identity: $x^2 = 0$, From (ii) it follows that $[\cdot, \cdot]$ is non-associative. Moreover, a Lie algebra is abelian if and only if $[\cdot, \cdot] = 0$.

Example

- Every vector space V is a Lie algebra taking $[\cdot, \cdot] = 0$.It is called the abelian Lie algebra.
- The cross product $(\mathbb{R}^3, \times)$ is a Lie algebra.

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Examples

- **The General Linear Lie Algebra $\mathfrak{gl}(V)$.** Let V an $\mathbb{F}$-vector space. Define:

$$[T, S] = T \circ S - S \circ T, \quad T, S \in \mathrm{End}(V).$$

Notation: $\mathfrak{gl}(V) = (\mathrm{End}(V), [\cdot, \cdot])$.

- **The Special Linear Lie Algebra $\mathfrak{sl}(V)$.**

$$\mathfrak{sl}(V) = \{T \in \mathrm{End}(V) : \mathrm{tr}(T) = 0\} \subset \mathfrak{gl}(V).$$

- Let $\mathcal{B} : V \times V \to \mathbb{F}$ be a bilinear form.

$$\mathfrak{gl}(V)_{\mathcal{B}} = \{T \in \mathrm{End}(V) : \mathcal{B}(T(u), v) + \mathcal{B}(u, T(v)) = 0\}$$

is a subalgebra of $\mathfrak{gl}(V)$.

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A morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ is a linear map such that

$$\phi[x, y]_{\mathfrak{g}} = [\phi(x), \phi(y)]_{\mathfrak{h}}, \quad x, y \in \mathfrak{g}.$$

The Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic if there exists a bijective morphism $\phi : \mathfrak{g} \to \mathfrak{h}$.

Example

- $(\mathbb{R}^3, \times) \simeq \mathfrak{gl}(R^2)_{\mathcal{B}}$, where $\mathcal{B} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Let V a vector space. A morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a representation.

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Associative ⇜⇝ Lie

Example

Let $(A, \cdot)$ be an associative algebra. Define a new product by

$$[a, b] = a \cdot b - b \cdot a, \;\; a, b \in A.$$

Then, $A^- = (A, [\cdot, \cdot])$ is a Lie algebra.

Theorem (Poincaré-Birkhoff-Witt)

Let $\mathfrak{g}$ be a Lie algebra, then $\mathfrak{g}$ is isomorphic to A^- for some associative algebra A.

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Structure Constants of a Lie Algebra

Let $\mathfrak{g}$ be a Lie algebra with product $[\cdot,\cdot]$, and let $\{e_1,\ldots,e_n\}$ be a basis for $\mathfrak{g}$.
Notice that:

$$[x,y]=\sum_{i,j} x_i y_j [e_i,e_j], \quad x,y\in\mathfrak{g}.$$

Write

$$[e_i,e_j]=\sum_{k=1}^{n} c_{ij}^{k} e_k. \quad i,j\in\{i,\ldots,n\}.$$

Definition

The set $\{c_{ij}^{k}\}\subset F$ is called the set of structure constants of $\mathfrak{g}$ respect to the basis $\{e_i\}_{i=1}^{n}$.

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$$[e_i,e_j]=\sum_{k=1}^{n}c_{ij}^{k}e_k.\quad i,j\in\{i,\ldots,n\}.$$

Definition

The set $\{c_{ij}^k\}\subset F$ is called the set of structure constants of $\mathfrak{g}$ respect to the basis $\{e_i\}_{i=1}^n$.

Structure Constants of a Lie Algebra

Let $\mathfrak{g}$ be a Lie algebra with product $[\cdot,\cdot]$, and let $\{e_1,\ldots,e_n\}$ be a basis for $\mathfrak{g}$. Notice that:

$$[x,y]=\sum_{i,j} x_i y_j [e_i,e_j], \quad x,y\in\mathfrak{g}.$$

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Structure Constants of a Lie Algebra

Remark. Notice that

- From the skew symmetry:

$$c_{ij}^k + c_{ji}^k = 0, \qquad \text{for all } i, j, k \in \{1, \ldots, n\}.$$

- From the Jacobi identity;

$$\sum_{l=1}^{n} c_{ij}^l c_{lk}^m + c_{ki}^l c_{lj}^m + c_{jk}^l c_{li}^m = 0,$$

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The Variety $\mathcal{L}ie_n$

If

(i) $P_1((x_{ij}^k)) = x_{ij}^k + x_{ji}^k$,

(ii) $P_2((x_{ij}^k)) = \sum_{l=1}^n x_{ij}^l x_{lk}^m + x_{ki}^l x_{lj}^m + x_{jk}^l x_{li}^m$,

then

$$\mathcal{L}ie_n = \mathcal{A}lg_n(P_1, P_2) \subset A^{n^3}$$

is a variety where every point represent a Lie algebra.

Structure Constants in general

Let V be an n-dimensional $\mathbb{F}$-vector space and let $\{e_1, \ldots, e_n\}$ be a fixed basis for V. Given an algebra $A = (V, \cdot)$ we write

$$e_i \cdot e_j = \sum_{k=1}^{n} x_{ij}^k e_k, \quad i, j = 1, \ldots, n,$$

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The Variety $\mathcal{A}lg_n(P_1, \ldots, P_r)$

Let $A = (V, \cdot)$ be an algebra defined by the polynomial identities $P_1, \ldots, P_r$. Then its set of structure constants must satisfy these identities.

$$\left\{ \begin{array}{c} \text{Algebras over } V \\ \text{satisfying } P_1, \ldots, P_r \end{array} \right\} \longleftrightarrow \mathcal{A}lg_n(P_1, \ldots, P_r) \subset \mathbb{A}^{n^3} \quad \text{affine variety}$$

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Change of basis

Given a $\mathbb{F}$-vector space V with fixed basis $\{e_i\}_{i=1}^n$, we have

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What happens if we change the basis of V?

Change of basis

Lemma

There is an action of $G = \mathrm{GL}(\mathbb{F}^n)$ *on* $\mathcal{L}ie_n$ *given by "change of basis"*

$$g \cdot [\cdot,\cdot] = [\cdot,\cdot]' = g[g^{-1}(\cdot), g^{-1}(\cdot)].$$

where

(i) *The G-orbits are in one-to-one correspondence with the isomorphism classes.*

(ii) $\mathrm{Stab}_G([\cdot,\cdot]) \longleftrightarrow \mathrm{Aut}([\cdot,\cdot])$.

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Interesting Problems

(Recall $G = GL(\mathbb{F}^n)$)
Given the action "change of basis"

$$G \times \mathcal{L}ie_n \to \mathcal{L}ie_n$$

(i) To find the G-orbits.
(ii) To find the Zariski closure of every G-orbit.

Definition
Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}ie_n$. We say that $\mathfrak{g}$ degenerates to $\mathfrak{h}$ ($\mathfrak{g} \to \mathfrak{h}$) if

$$\mathfrak{h} \in \overline{G \cdot \mathfrak{g}}$$

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Consider the action "change of basis"

$$\mathrm{GL}(\mathbb{F}^n) \times \mathcal{L}ie_n \to \mathcal{L}ie_n$$

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Example: Lie Algebras Case with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

Consider the action "change of basis"

$$GL(V) \times \mathcal{L}ie_n \to \mathcal{L}ie_n$$

- For $n = 3$ there exist an infinite number of orbits: $\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3$ with:

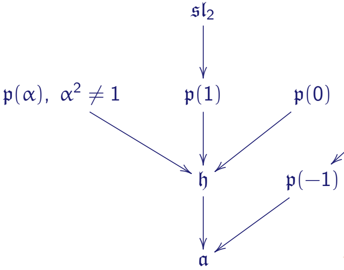
	$\mathfrak{sl}_2$	$\mathfrak{su}_2$	$\mathfrak{r}$	$\mathfrak{q}(\beta)$	$\mathfrak{p}(\alpha)$	$\mathfrak{h}$	$\mathfrak{a}$
$[e_2, e_3]$	e_1	e_1	$-(e_1 + e_3)$	$e_1 - \beta e_3$	αe_3	0	0
$[e_3, e_1]$	e_2	e_2	0	0	0	0	0
$[e_1, e_2]$	e_3	$-e_3$	e_1	$\beta e_1 + e_3$	e_1	e_3	0

with $\alpha, \beta \in \mathbb{R}$. Over $\mathbb{C}$: $\mathfrak{sl}_2 \simeq \mathfrak{su}_2$ and $\mathfrak{r} \simeq \mathfrak{q}(\beta)$.

Example: Lie Algebras with $\mathbb{F} = \mathbb{C}$

[D. Burde and C. Steinhoff, 1998]
For $n = 3$.

	$\mathfrak{sl}_2$	$\mathfrak{r}$	$\mathfrak{p}(\alpha)$	$\mathfrak{h}$	$\mathfrak{a}$
$[e_2, e_3]$	e_1	$-(e_1 + e_3)$	αe_3	0	0
$[e_3, e_1]$	e_2	0	0	0	0
$[e_1, e_2]$	e_3	e_1	e_1	e_3	0



The Lie superalgebra case

Supervector spaces

Super $= \mathbb{Z}_2$-graded.

Definition

A supervector space is a vector space $V = V_0 \oplus V_1$.

- The subspaces V_0 and V_1 are called the even and the odd parts of V, resp.
- The elements in $(V_0 \cup V_1) \setminus \{0\}$ are called homogeneous.
- The degree of a homogeneous element is defined as

$$|x| = i, \quad \text{if } x \in V_i \setminus \{0\}.$$

- If $\dim(V_0) = m$ and $\dim(V_1) = n$ we say that

$$\dim(V) = (m, n).$$

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Superalgebras

Definition

A superalgebra $A = A_0 \oplus A_1$ is a supervector space with a bilinear map $A \times A \to A$ such that

$$A_i A_j \subset A_{i+j}, \quad i, j \in \mathbb{Z}_2.$$

Remark

(i) A_0 is a subalgebra.

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Example

Let $V = V_0 \oplus V_1$ be a supervector space. Let $T : V \to V$,

$$\mathrm{End}(V_0|V_1)_0 \ni T \iff T = X \oplus Y \leftrightarrow \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad \text{where} \quad \begin{matrix} X : V_0 \to V_0 \\ Y : V_1 \to V_1 \end{matrix}$$

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Then

$$\mathrm{End}(V_0|V_1) := \mathrm{End}(V) = \mathrm{End}(V_0|V_1)_0 \oplus \mathrm{End}(V_0|V_1)_1,$$

and $\mathrm{End}(V_0|V_1)$ is a superalgebra with the composition of functions as product.

Superalgebras

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$$\mathrm{End}(V_0|V_1) := \mathrm{End}(V) = \mathrm{End}(V_0|V_1)_0 \oplus \mathrm{End}(V_0|V_1)_1,$$

and $\mathrm{End}(V_0|V_1)$ is a superalgebra with the composition of functions as product.

Lie Superalgebras

Definition

A Lie superalgebra is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with product $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, where the homogeneous elements satisfy:

(i) The super skew-symmetry

$$[\![x, y]\!] = -(-1)^{|x||y|}[\![y, x]\!].$$

(ii) The super Jacobi identity

$$(-1)^{|x||z|}[\![[\![x, y]\!], z]\!] + (-1)^{[x][y]}[\![[\![y, z]\!], x]\!] + (-1)^{[y][z]}[\![[\![z, x]\!], y]\!] = 0,$$

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Lie Superalgebras

Example

Let $V = V_0 \oplus V_1$ be a supervector space, and let $T, S \in \mathrm{End}(V_0|V_1)$ homogeneous. Define:

$$[\![T, S]\!] = T \circ S - (-1)^{|T||S|} S \circ T.$$

Then, $\mathfrak{gl}(V_0|V_1) := (\mathrm{End}(V_0|V_1), [\![\cdot, \cdot]\!])$ is a Lie superalgebra.

Definition

Let $T \in \mathrm{End}(V_0|V_1)$. Define the supertrace as follows

$$\text{if} \quad T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \quad \text{then} \quad str(T) = tr(X) - tr(W).$$

Example

The superalgebra $\mathfrak{sl}(V_0|V_1) = \{T \in \mathrm{End}(V_0|V_1) | str(T) = 0\} \subset \mathfrak{gl}(V_0|V_1)$

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Morphisms

Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}' = \mathfrak{g}'_0 \to \mathfrak{g}'_1$ be Lie superalgebras. A Lie superalgebra morphism $\Phi : \mathfrak{g} \to \mathfrak{g}'$ is a linear map such that

$$\Phi(\llbracket \cdot, \cdot \rrbracket_{\mathfrak{g}}) = \llbracket \Phi(\cdot), \Phi(\cdot) \rrbracket_{\mathfrak{g}'}$$

Remark

- Notice that Φ is an even map, i.e.,

$$\Phi(\mathfrak{g}_0) \subset \mathfrak{g}'_0 \quad \text{and} \quad \Phi(\mathfrak{g}_1) \subset \mathfrak{g}'_1.$$

- We can write $\Phi = T \oplus S$, where $T : \mathfrak{g}_0 \to \mathfrak{g}'_0$ is a Lie algebra morphism, and $S : \mathfrak{g}_1 \to \mathfrak{g}'_1$ is a $\mathfrak{g}_0$-module morphism such that

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The Variety $\mathcal{LS}^{(m,n)}$

Let $V = V_0 \oplus V_1$ be a complex $(m,n)-$dimensional supervector space with a fixed homogeneous basis $\{e_1, \dots, e_m, f_1, \dots, f_n\}$. Given a Lie superalgebra structure on V, we identify $\mathfrak{g} = (V, [\![\cdot,\cdot]\!])$ with its set of structure constants

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \longleftrightarrow (c_{ij}^k, \rho_{ij}^k, \Gamma_{ij}^k) \in \mathbb{C}^{m^3+2mn^2}$$

where

$$[\![e_i, e_j]\!] = \sum_{k=1}^{m} c_{ij}^k e_k, \quad [\![e_i, f_j]\!] = \sum_{k=1}^{n} \rho_{ij}^k f_k, \quad \text{and} \quad [\![f_i, f_j]\!] = \sum_{k=1}^{m} \Gamma_{ij}^k e_k.$$

Notation

$\mathcal{LS}^{(m,n)}$ denotes the Lie superalgebra variety of dimension (m,n) over $\mathbb{C}$.

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The Action by "Change of Basis"

Let $G = GL_m(\mathbb{C}) \oplus GL_n(\mathbb{C})$. We have the action $G \times \mathcal{LS}^{(m,n)} \to \mathcal{LS}^{(m,n)}$ given by

$$g \cdot [\![\cdot, \cdot]\!] = g[\![g^{-1}(\cdot),\, g^{-1}(\cdot)]\!]$$

An Amazing Dream

To find:

- the G-orbits,
- the Zariski closure of every G-orbit,

for

$$\mathcal{LS}^{(m,n)}, \quad m, n \in \mathbb{N}.$$

The reality

We studied the problem for

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The G-orbits of $\mathcal{LS}^{(2,2)}$

Theorem (Alvarez,M.A,–)

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a complex Lie superálgebra of dimension $(2,2)$, then $\mathfrak{g}$ is isomorphic to one and only one of the following:

$$
\begin{array}{llll}
\mathcal{LS}_0: & [\![\cdot,\cdot]\!] = 0. & & \\
\mathcal{LS}_1: & [\![f_1,f_1]\!] = e_1, & [\![f_2,f_2]\!] = e_2. & \\
\mathcal{LS}_2: & [\![f_1,f_1]\!] = e_1, & [\![f_2,f_2]\!] = e_1. & \\
\mathcal{LS}_3: & [\![f_1,f_1]\!] = e_1. & & \\
\mathcal{LS}_4: & [\![f_1,f_2]\!] = e_1, & [\![f_2,f_2]\!] = e_2. & \\
\mathcal{LS}_5: & [\![e_1,f_1]\!] = f_1, & [\![e_2,f_2]\!] = f_2. & \\
\mathcal{LS}_6(\alpha): & [\![e_2,f_1]\!] = f_1, & [\![e_2,f_2]\!] = \alpha f_2. & \\
\mathcal{LS}_7: & [\![e_2,f_1]\!] = f_1, & [\![e_2,f_2]\!] = -f_2, & [\![f_1,f_2]\!] = e_1. \\
\mathcal{LS}_8: & [\![e_2,f_1]\!] = f_1, & [\![f_2,f_2]\!] = e_1. & \\
\mathcal{LS}_9: & [\![e_1,f_1]\!] = f_1, & [\![e_1,f_2]\!] = f_2, & [\![e_2,f_2]\!] = f_1. \\
\mathcal{LS}_{10}: & [\![e_2,f_1]\!] = f_1, & [\![e_2,f_2]\!] = f_1 + f_2. & \\
\mathcal{LS}_{11}: & [\![e_2,f_2]\!] = f_1. & & \\
\mathcal{LS}_{12}: & [\![e_2,f_2]\!] = f_1, & [\![f_2,f_2]\!] = e_1. &
\end{array}
$$

The G-orbits of $\mathcal{LS}^{(2,2)}$

$\mathcal{LS}_{13}(\alpha,\beta)$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_2,f_1]\!]=\alpha f_1,$	$[\![e_2,f_2]\!]=\beta f_2.$
$\mathcal{LS}_{14}(\alpha)$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_2,f_1]\!]=\alpha f_1,$	$[\![e_2,f_2]\!]=-(\alpha+1)f_2,$
	$[\![f_1,f_2]\!]=e_1,$	$\alpha\neq-\frac{1}{2}.$	
$\mathcal{LS}_{15}(\alpha)$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_2,f_1]\!]=\alpha f_1,$	$[\![e_2,f_2]\!]=-\frac{1}{2}f_2,$
	$[\![f_1,f_1]\!]=\delta_{-\frac{1}{2},\alpha}e_1,$	$[\![f_2,f_2]\!]=e_1.$	
$\mathcal{LS}_{16}$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_2,f_1]\!]=-\frac{1}{2}f_1,$	$[\![e_2,f_2]\!]=-\frac{1}{2}f_2,$
	$[\![f_1,f_1]\!]=e_1.$		
$\mathcal{LS}_{17}(\alpha)$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_2,f_1]\!]=\alpha f_1,$	$[\![e_2,f_2]\!]=f_1+\alpha f_2,$
	$[\![f_2,f_2]\!]=\delta_{-\frac{1}{2},\alpha}e_1.$		
$\mathcal{LS}_{18}$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_2,f_1]\!]=-\frac{1}{2}f_1,$	$[\![e_2,f_2]\!]=f_1-\frac{1}{2}f_2.$
$\mathcal{LS}_{19}(\alpha)$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_1,f_2]\!]=f_1,$	$[\![e_2,f_1]\!]=\alpha f_1,$
	$[\![e_2,f_2]\!]=(\alpha+1)f_2,$	$[\![f_1,f_2]\!]=\delta_{-1,\alpha}e_1,$	$[\![f_2,f_2]\!]=2\delta_{-1,\alpha}e_2.$
$\mathcal{LS}_{20}$:	$[\![e_1,e_2]\!]=e_1,$	$[\![e_1,f_2]\!]=f_1,$	$[\![e_2,f_1]\!]=-f_1,$

where $\alpha,\beta\in\mathbb{C}$.

- $\mathcal{LS}_n(\alpha)\simeq\mathcal{LS}_n(\alpha')\Leftrightarrow\alpha=\alpha'$, for $n\in\{6,15,17,19\}$
- $\mathcal{LS}_{14}(\alpha)\simeq\mathcal{LS}_{14}(\alpha')\Leftrightarrow$ either $\alpha=\alpha'$ or $\alpha+\alpha'=-1$.
- $\mathcal{LS}_{13}(\alpha,\beta)\simeq\mathcal{LS}_{13}(\alpha',\beta')\Leftrightarrow\{\alpha,\beta\}=\{\alpha',\beta'\}$.

Next Step

To find the orbit closure for each Lie superalgebra $\mathcal{LS}_n$, for $n = 0, \cdots, 20$.

Degenerations in $\mathcal{LS}^{(m,n)}$

Definition

Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{LS}^{(m,n)}$. We say that $\mathfrak{g}$ degenerates to $\mathfrak{h}$ (denoted by $\mathfrak{g} \to \mathfrak{h}$) if

$$\mathfrak{h} \in \overline{G \cdot \mathfrak{g}} \quad \text{(Zariski closure)}$$

Remark

Notice that

$$\text{if} \quad \mathfrak{g} \to \mathfrak{h} \quad \text{and} \quad \mathfrak{h} \to \mathfrak{s}, \quad \text{then} \quad \mathfrak{g} \to \mathfrak{s}.$$

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Lemma

Let $\mathbb{C}(t)$ *be the field of fractions of the polynomial ring* $\mathbb{C}[t]$*, and let* $\mathfrak{g}, \mathfrak{h} \in \mathcal{LS}^{(m,n)}$*. If there exists a continuous map*

$$(0,1] \to \mathrm{GL}_m(\mathbb{C}(t)) \oplus \mathrm{GL}_n(\mathbb{C}(t)), \quad t \mapsto g_t$$

such that $\lim_{t\to 0} g_t \cdot \mathfrak{g} = \mathfrak{h}$*, then* $\mathfrak{g} \to \mathfrak{h}$*.*

Remark

Let $\mathfrak{g} \in \mathcal{LS}^{(m,n)}$, then $\mathfrak{g}$ degenerates to the trivial Lie superalgebra $\mathfrak{a} = (0,0,0)$ (take $t \to t^{-1}(\mathrm{id}_m \oplus \mathrm{id}_n)$.)

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Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \in \mathcal{LS}^{(m,n)}$. *If* $\mathfrak{g} \to \mathfrak{h}$, *then* $\mathfrak{g}_0 \to \mathfrak{h}_0$.

Degenerations in $\mathcal{LS}^{(m,n)}$

Definition

Let $\mathfrak{g} = \left\{c_{ij}^k, \rho_{ij}^k, \Gamma_{ij}^k\right\} \in \mathcal{LS}^{(m,n)}$.

(i) The **abelianization of** $\mathfrak{g}$ is defined as $ab(\mathfrak{g}) = \left\{0, 0, \Gamma_{ij}^k\right\}$.

(ii) The **forgetful Lie superalgebra of** $\mathfrak{g}$ is defined as $\mathcal{F}(\mathfrak{g}) = \left\{c_{ij}^k, \rho_{ij}^k, 0\right\}$.

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Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{LS}^{(m,n)}$. *If* $\mathfrak{g} \to \mathfrak{h}$, *then*

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Invariants in $\mathcal{LS}^{(m,n)}$

Lemma

Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{LS}^{(m,n)}$. If $\mathfrak{g} \to \mathfrak{h}$, then the following relations must hold:

- $\dim(G \cdot \mathfrak{g}) > \dim(G \cdot \mathfrak{h})$

  ($\dim(G \cdot \mathfrak{g}) = \dim(G) - \dim(Der(\mathfrak{g}))$).

- $\dim(\mathfrak{g}^1)_i \geqslant \dim(\mathfrak{h}^1)_i$ *for* $i \in \mathbb{Z}_2$.

- *If* $[\![\cdot,\cdot]\!]_{\mathfrak{g}_1 \times \mathfrak{g}_1} \equiv 0$, *then* $[\![\cdot,\cdot]\!]_{\mathfrak{h}_1 \times \mathfrak{h}_1} \equiv 0$.

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Invariants in $\mathcal{LS}^{(m,n)}$

- $\dim(Der(\alpha, \beta, \gamma))_i(\mathfrak{g}) \leqslant \dim(Der(\alpha, \beta, \gamma))_i(\mathfrak{h})$, for $i \in \mathbb{Z}_2$.

  ($D \in Der(\alpha, \beta, \gamma)(\mathfrak{g})_i$ if $\alpha D([\![x, y]\!]) = \beta[\![D(x), y]\!] + (-1)^{i|x|}\gamma[\![x, D(y)]\!]$).

The Variety $\mathcal{LS}^{(2,2)}$

Lie algebras of dimension 2. There exist, up to isomorphism, two Lie algebras of dimension 2.

- The abelian Lie algebra $\mathfrak{a}_0$.
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It follows that

$$\mathfrak{a}_1 \to \mathfrak{a}_0$$

Remark

Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{LS}^{(m,n)}$ such that $\mathfrak{g}_0 = \mathfrak{a}_0$, and $\mathfrak{h}_0 = \mathfrak{a}_1$, then

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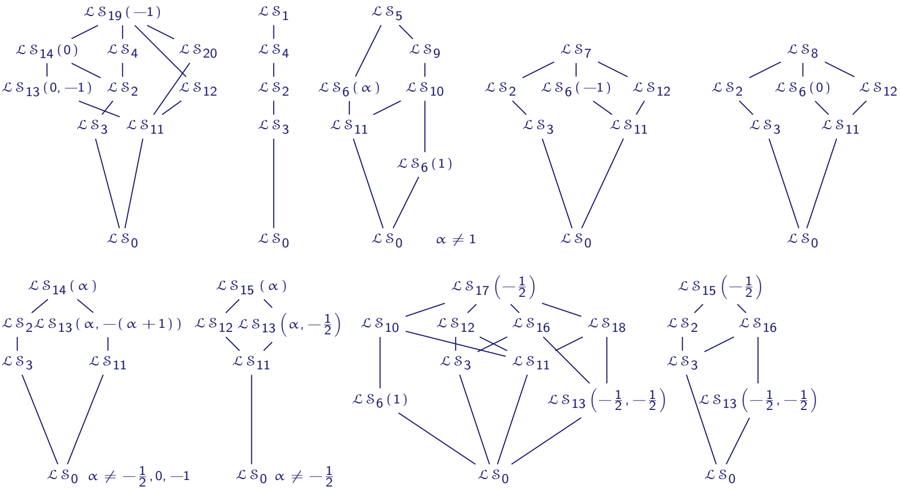
Hasse Diagrams in $\mathcal{LS}^{(2,2)}$



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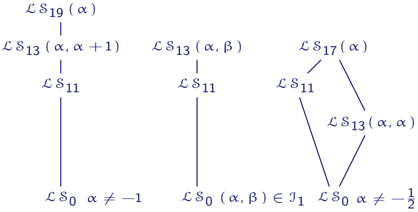

where $\mathcal{I}_1 = \left\{ (\alpha, \beta) \,\middle|\, \begin{array}{l} \alpha \neq -\frac{1}{2},\ \beta,\ \beta+1,\ -(\beta+1), \text{ and} \\ \beta \neq -\frac{1}{2},\ \alpha,\ \alpha+1,\ -(\alpha+1) \end{array} \right\}.$

Orbits in $\mathcal{LS}^{(2,2)}$

The orbit closures in the variety $\mathcal{LS}^{(2,2)}$ are as follows:

$\mathfrak{g}$	$\overline{G \cdot \mathfrak{g}}$
$\mathcal{LS}_0$	$\mathcal{LS}_0$
$\mathcal{LS}_1$	$\mathcal{LS}_1, \mathcal{LS}_2, \mathcal{LS}_3, \mathcal{LS}_4, \mathcal{LS}_0$
$\mathcal{LS}_2$	$\mathcal{LS}_2, \mathcal{LS}_3, \mathcal{LS}_0$
$\mathcal{LS}_3$	$\mathcal{LS}_3, \mathcal{LS}_0$
$\mathcal{LS}_4$	$\mathcal{LS}_4, \mathcal{LS}_2, \mathcal{LS}_3, \mathcal{LS}_0$
$\mathcal{LS}_5$	$\mathcal{LS}_5, \mathcal{LS}_9, \mathcal{LS}_6(\alpha), \mathcal{LS}_{10}, \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_6(\alpha), \ \alpha \neq 1$	$\mathcal{LS}_6(\alpha), \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_6(1)$	$\mathcal{LS}_6(1), \mathcal{LS}_0$
$\mathcal{LS}_7$	$\mathcal{LS}_7, \mathcal{LS}_2, \mathcal{LS}_6(-1), \mathcal{LS}_{12}, \mathcal{LS}_3, \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_8$	$\mathcal{LS}_8, \mathcal{LS}_2, \mathcal{LS}_6(0), \mathcal{LS}_{12}, \mathcal{LS}_3, \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_9$	$\mathcal{LS}_9, \mathcal{LS}_{10}, \mathcal{LS}_{11}, \mathcal{LS}_6(1), \mathcal{LS}_0$
$\mathcal{LS}_{10}$	$\mathcal{LS}_{10}, \mathcal{LS}_{11}, \mathcal{LS}_6(1), \mathcal{LS}_0$
$\mathcal{LS}_{11}$	$\mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_{12}$	$\mathcal{LS}_{12}, \mathcal{LS}_3, \mathcal{LS}_{11}, \mathcal{LS}_0$

Orbits in $\mathcal{LS}^{(2,2)}$

$\mathcal{LS}_{13}(\alpha,\beta),\ \ \beta\neq\alpha$	$\mathcal{LS}_{13}(\alpha,\beta), \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_{13}(\alpha,\alpha)$	$\mathcal{LS}_{13}(\alpha,\alpha), \mathcal{LS}_0$
$\mathcal{LS}_{14}(\alpha),\ \ \alpha\neq-\frac{1}{2}$	$\mathcal{LS}_{14}(\alpha), \mathcal{LS}_2, \mathcal{LS}_{13}(\alpha,-(\alpha+1)), \mathcal{LS}_3, \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_{15}(\alpha),\ \ \alpha\neq-\frac{1}{2}$	$\mathcal{LS}_{15}(\alpha), \mathcal{LS}_{12}, \mathcal{LS}_{13}\left(\alpha,-\frac{1}{2}\right), \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_{15}\left(-\frac{1}{2}\right)$	$\mathcal{LS}_{15}\left(-\frac{1}{2}\right), \mathcal{LS}_2, \mathcal{LS}_{16}, \mathcal{LS}_3, \mathcal{LS}_{13}\left(-\frac{1}{2},-\frac{1}{2}\right), \mathcal{LS}_0$
$\mathcal{LS}_{16}$	$\mathcal{LS}_{16}, \mathcal{LS}_3, \mathcal{LS}_{13}\left(-\frac{1}{2},-\frac{1}{2}\right), \mathcal{LS}_0$
$\mathcal{LS}_{17}(\alpha),\ \ \alpha\neq-\frac{1}{2}$	$\mathcal{LS}_{17}(\alpha), \mathcal{LS}_{11}, \mathcal{LS}_{13}(\alpha,\alpha), \mathcal{LS}_0$
$\mathcal{LS}_{17}\left(-\frac{1}{2}\right)$	$\mathcal{LS}_{17}\left(-\frac{1}{2}\right), \mathcal{LS}_{10}, \mathcal{LS}_{12}, \mathcal{LS}_{16}, \mathcal{LS}_{18}, \mathcal{LS}_3,$ $\mathcal{LS}_{11}, \mathcal{LS}_6(1), \mathcal{LS}_{13}\left(-\frac{1}{2},-\frac{1}{2}\right), \mathcal{LS}_0$
$\mathcal{LS}_{18}$	$\mathcal{LS}_{18}, \mathcal{LS}_{11}, \mathcal{LS}_{13}\left(-\frac{1}{2},-\frac{1}{2}\right), \mathcal{LS}_0$
$\mathcal{LS}_{19}(\alpha),\ \ \alpha\neq-1$	$\mathcal{LS}_{19}(\alpha), \mathcal{LS}_{13}(\alpha,\alpha+1), \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_{19}(-1)$	$\mathcal{LS}_{19}(-1), \mathcal{LS}_4, \mathcal{LS}_{20}, \mathcal{LS}_2, \mathcal{LS}_{12}, \mathcal{LS}_3, \mathcal{LS}_{11}, \mathcal{LS}_0$
$\mathcal{LS}_{20}$	$\mathcal{LS}_{20}, \mathcal{LS}_{11}, \mathcal{LS}_0$

Muito obrigada!

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The irreducible components of $\mathcal{LS}^{(2,2)}$

Lemma

If G is a connected algebraic group acting on a variety X, then the irreducible components of X are stable under the action of G. Moreover, the irreducible components of the variety X are closures of single orbits or closures of infinite families of orbits.

Irreducible components of $\mathcal{LS}^{(2,2)}$

Definition

A Lie superalgebra $\mathfrak{g} \in \mathcal{LS}^{(m,n)}$ is called rigid, if its orbit $G \cdot \mathfrak{g}$ is open in $\mathcal{LS}^{(m,n)}$.

Lemma

Let $\mathfrak{g} \in \mathcal{LS}^{(m,n)}$. *If* $\mathrm{H}^2(\mathfrak{g}, \mathfrak{g})_0 = 0$, *then* $\mathfrak{g}$ *is rigid*

Remark

Every rigid superalgebra $\mathfrak{g}$ determines an irreducible component of $\mathcal{LS}^{(m,n)}$.

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There are 3 rigid Lie superalgebras in $\mathcal{LS}^{(2,2)}$

$$\mathcal{LS}_{19}(-1), \quad \mathcal{LS}_1, \quad \mathcal{LS}_5.$$

Corollary

The variety $\mathcal{LS}^{(2,2)}$ *has at least four irreducible components.*

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The irreducible components of $\mathcal{LS}^{(2,2)}$

Conjecture

The irreducible components of the variety $\mathcal{LS}^{(2,2)}$ are:

(i) $\mathcal{C}_1 = \overline{G \cdot \mathcal{LS}_{19}(-1)}$,

(ii) $\mathcal{C}_2 = \overline{G \cdot \mathcal{LS}_1}$,

(iii) $\mathcal{C}_3 = \overline{G \cdot \mathcal{LS}_5}$,

(iv) $\mathcal{C}_4 = \overline{\bigcup_{\alpha \neq -\frac{1}{2}} G \cdot \mathcal{LS}_{14}(\alpha)}$,

(v) $\mathcal{C}_5 = \overline{\bigcup_{\alpha \neq -\frac{1}{2}} \mathcal{LS}_{15}(\alpha)}$,

(vi) $\mathcal{C}_6 = \overline{\bigcup_{\alpha \neq -1} \mathcal{LS}_{19}(\alpha)}$,

(vii) $\mathcal{C}_7 = \overline{\bigcup_{\alpha \beta} \mathcal{LS}_{13}(\alpha, \beta)}$.