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Time reversed Markov jump linear quadratic problem and applications

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Introduction

Let the parameters A, B, C, D, E, e F of appropriate dimensions. We define a liner system (LS), as follows:

$$\left\{ \begin{array}{ccc} x(t+1) & = & Ax(t) + Bu(t) + Ew(t), \\ y(t) & = & Cx(t) + Du(t) + Fw(t), \\ x(0) & = & x_0. \end{array} \right.$$

We denote a LS by (A, B, C, D, E, F).

Introduction

Given a initial state $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ and consider a dynamic of $x(t)$ by

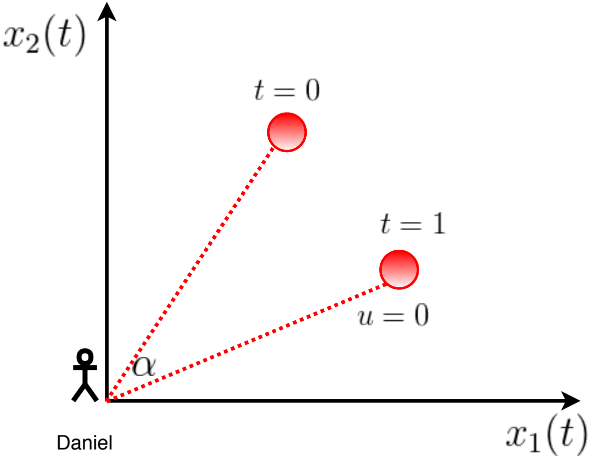
$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \mathrm{sen}(\alpha) \\ -\mathrm{sen}(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \; \forall\, t = 0, 1, \ldots .$$

Also,

- The control is inactive, when $u(t) = 0$.
- The control is active, when $u(t) = 1$.

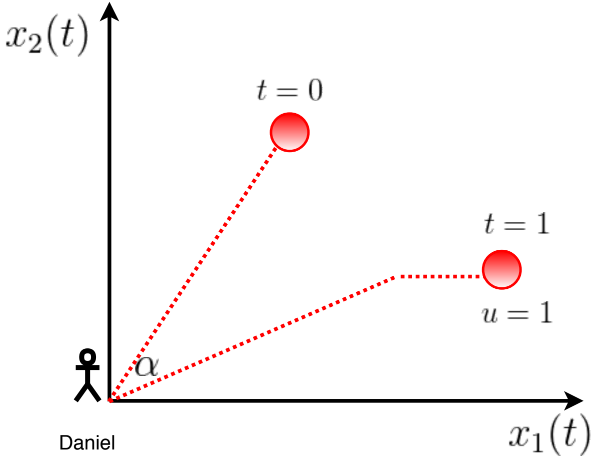
Introduction

Inactive control



Introduction

Active control



Introduction

Given a LS,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \ \forall\, t = 0, 1, \ldots, \ell - 1, \end{aligned}$$

with the conditions $C'D = 0$ and $D'D > 0$.

Linear-quadratic regulator

Obtain $u^{\mathrm{opt}}(0), u^{\mathrm{opt}}(1), \ldots, u^{\mathrm{opt}}(\ell)$ that minimize

$$V = \sum_{t=0}^{\ell-1} \|y(t)\|^2 + x'(\ell)\Gamma x(\ell),$$

Introduction

Using dynamic programming, we obtain $u^{\text{opt}}(t) = K(t)x(t)$ where

$$K(t) = -\big(B'P(t+1)B + D'D\big)^{-1} B'P(t+1)A.$$

Also, $V = x'(0)P(0)x(0)$.

Difference Riccati equation

Let $P(\ell) = \Gamma$. For all $t = \ell, \ell - 1, \ldots, 1$, compute

$$P(t-1) = C'C + A'P(t)A - A'P(t)B\big(B'P(t)B + D'D\big)^{-1} B'P(t)A.$$

Markov jump linear systems

Let a Markov chain $\theta(t) \in \{1, 2, \dots, N\}$ with transition probability, given by

$$p_{ij} = \text{Prob}\big(\theta(t+1) = j \,|\, \theta(t) = i\big), \ \text{ for each } i, j = 1, \dots, N.$$

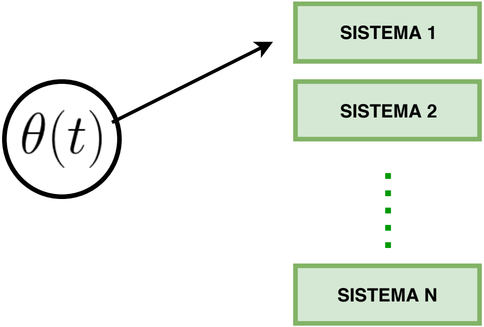
and probability distribution at time t,

$$\pi(t) = [\pi_1(t), \dots, \pi_N(t)]\,,$$

where $\pi_i(t) = \text{Prob}\big(\theta(t) = i\big)$, for each $i = 1, \dots, N$.

Markov jump linear systems

For each time t, we have



Markov jump linear systems

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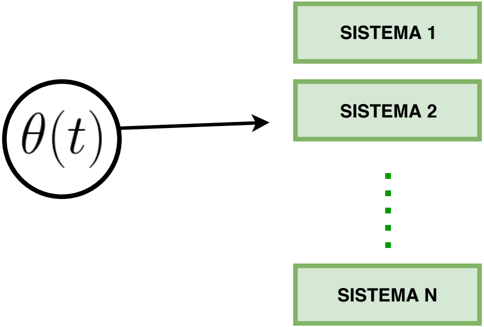
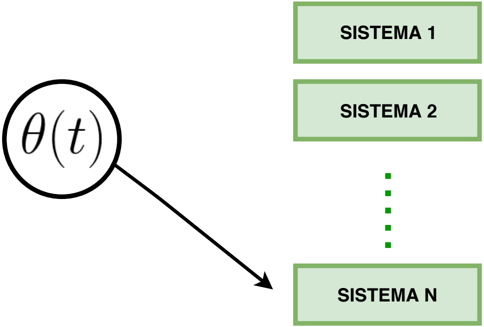


Markov jump linear systems

For each time t, we have



Markov jump linear systems

Diferently of LS, in this case we consider

$$
\begin{aligned}
\mathbb{A} &= \big(A_1, A_2, \ldots, A_N\big),\\
\mathbb{B} &= \big(B_1, B_2, \ldots, B_N\big),\\
\mathbb{C} &= \big(C_1, C_2, \ldots, C_N\big),\\
\mathbb{D} &= \big(D_1, D_2, \ldots, D_N\big),\\
\mathbb{E} &= \big(E_1, E_2, \ldots, E_N\big),\\
\mathbb{F} &= \big(F_1, F_2, \ldots, F_N\big).
\end{aligned}
$$

We define a MJLS, by

$$
\left\{
\begin{array}{ccl}
x(t+1) & = & A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t),\\
y(t) & = & C_{\theta(t)}x(t) + D_{\theta(t)}u(t) + F_{\theta(t)}w(t),\\
x(0) & = & x_0 \ \text{ e } \ \theta(0) \ = \ \theta_0.
\end{array}
\right.
$$

Also, we denote the above system by $\big(\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}\big)$.

Markov jump linear systems

Given an initial condition $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ e $\theta(t) \in \{1, 2, 3\}$. For each $t = 0, 1, \ldots$ define

- If $\theta(t) = 1$ follows

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

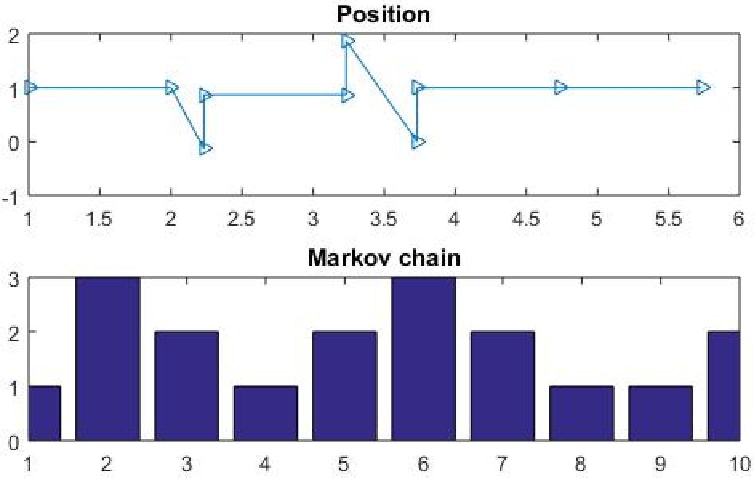
- If $\theta(t) = 2$ follows

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

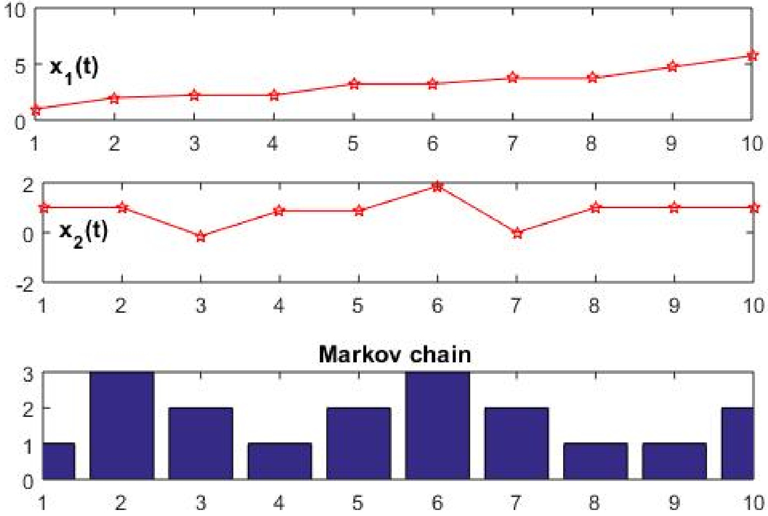
- If $\theta(t) = 3$ follows

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & \mathrm{sen}(\pi/6) \\ -\mathrm{sen}(\pi/6) & \cos(\pi/6) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t).$$

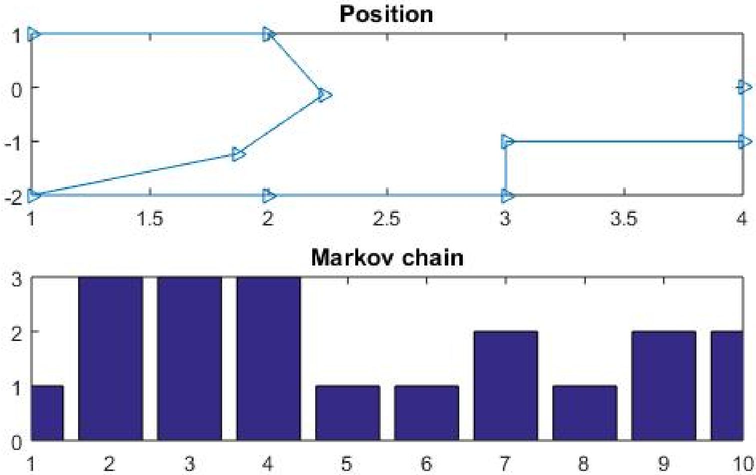
Markov jump linear systems



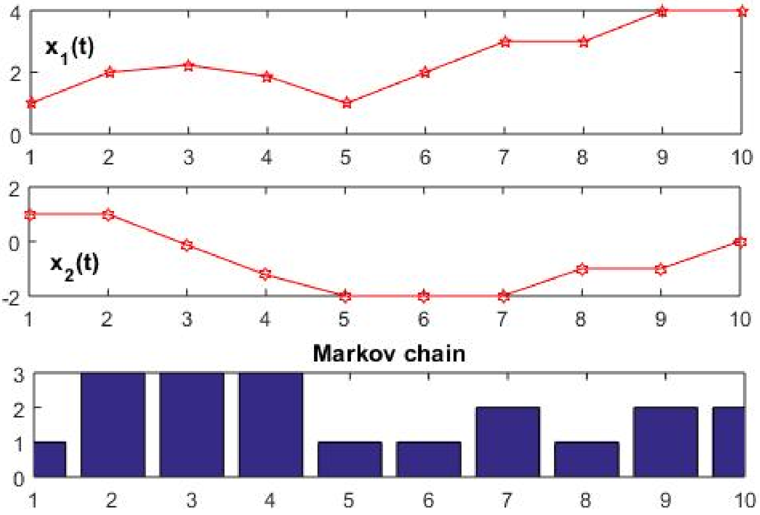
Markov jump linear systems



Markov jump linear systems



Markov jump linear systems



Markov jump linear systems

Considering $C_i'D_i = 0$ e $D_i'D_i > 0$ for each $i = 1, \ldots, N$, we define

$$\begin{aligned} x(t+1) &= A_{\theta(t)}x(t) + B_{\theta(t)}u(t), \\ y(t) &= C_{\theta(t)}x(t) + D_{\theta(t)}u(t), \;\; t = 0, 1, \ldots, \ell - 1. \end{aligned}$$

Jump linear-quadratic problem

Obtain $u^{\text{opt}}(0), u^{\text{opt}}(1), \ldots, u^{\text{opt}}(\ell)$ that minimize

$$V = \sum_{t=0}^{\ell-1} \mathbb{E}\{\|y(t)\|^2\} + \mathbb{E}\{x'(\ell)\Gamma x(\ell)\},$$

Markov jump linear systems

Suposse that

$$u(t) = K_{\theta(t)}(t)x(t), \ \forall\, t = 0, \ldots, \ell.$$

Then, for all $i = 1, \ldots, N$ we have

$$K_i(t) = -\big(B_i'\mathcal{E}_i(\mathbb{P}(t+1))B_i + D_i'D_i\big)^{-1} B_i'\mathcal{E}_i(\mathbb{P}(t+1))A_i,$$

where $\mathcal{E}(\mathbb{Y}) = \big(\mathcal{E}_1(\mathbb{Y}), \ldots, \mathcal{E}_N(\mathbb{Y})\big)$, $\mathbb{Y} = \big(Y_1, \ldots, Y_N\big)$ e

$$\mathcal{E}_i(\mathbb{Y}) = \sum_{j=1}^{N} p_{ij} Y_j, \ \forall\, i = 1, \ldots, N.$$

Markov jump linear systems

Given $\mathbb{Y} = \big(Y_1, \ldots, Y_N\big)$ and $\mathbb{Z} = \big(Z_1, \ldots, Z_N\big)$, we define the inner product, as follows

$$\langle \mathbb{Y}, \mathbb{Z} \rangle = \sum_{i=1}^{N} \text{Traço}(Y_i' Z_i).$$

Also, we define the moment variables $\mathbb{X}(t) = \big(X_1(t), \ldots, X_N(t)\big)$ whose elements are $X_i(t) = \mathbb{E}\{x(t)x'(t) \cdot \mathbf{1}_{\{\theta(t)=i\}}\}$, for each $i = 1, \ldots, N$, and calculate

$$V = \langle \mathbb{P}(0), \mathbb{X}(0) \rangle.$$

Markov jump linear systems

Coupled difference Riccati equations of control (CDRE-C)

Let $P_i(\ell) = \Gamma$, $\forall\, i = 1, \dots, N$. For all $t = \ell, \ell - 1, \dots, 1$ compute

$$P_i(t-1) = C_i'C_i + A'\mathcal{E}_i(\mathbb{P}(t))A_i - A_i'\mathcal{E}_i(\mathbb{P}(t))B_i \\ \cdot \left(B_i'\mathcal{E}_i(\mathbb{P}(t))B_i + D_i'D_i\right)^{-1} B_i'\mathcal{E}_i(\mathbb{P}(t))(t)A_i,\ \forall\, i = 1, \dots, N.$$

Coupled difference Riccati equations of filtering (CDRE-F)

Let $Q_i(0) = \Sigma$, $\forall\, i = 1, \dots, N$. For all $t = 0, 1, \dots, \ell - 1$ compute: if $\pi_i(t+1) = 0$ then $Q_i(t+1) = 0$, otherwise

$$Q_i(t+1) = \sum_{\{j\,:\,\pi_j(t)>0\}} p_{ji}\Big\{\pi_j(t)G_jG_j' + A_jQ_j(t)A_j' - A_jQ_j(t)L_j' \\ \cdot \left(L_jQ_j(t)L_j' + \pi_j(t)H_jH_j'\right)^{-1} L_jQ_j(t)A_j'\Big\}.$$

Time reversed Markov jump linear systems

Given a Markov chain θ and time horizon ℓ, we define the *time reversed Markov chain* (denoted by η), as follows:

$$\eta(t) = \theta(\ell - t), \ \forall\, t = 0, 1, \ldots, \ell.$$

Considering a Time reversed Markov jump linear system, (TRMJLS) as follows

$$\begin{cases} x(t+1) & = & A_{\eta(t)}x(t) + B_{\eta(t)}u(t) + e_{\eta(t)}(t), \ \forall\, t = 0, 1, \ldots, \ell - 1, \\ x(0) & = & x_0. \end{cases}$$

Time reversed Markov jump linear systems

Differently of MJLS, in this case, we use conditioned moment variables.

- We denote by $\mathfrak{m}(t) = (\mathfrak{m}_1(t), \ldots, \mathfrak{m}_N(t))$, the *conditioned first moment* whose elements are given by

$$\mathfrak{m}_i(t) = \mathbb{E}\big\{x(t) \,|\, \eta(t) = i\big\}, \ \forall\, i = 1, \ldots, N.$$

- We denote by $\mathfrak{M}(t) = (\mathfrak{M}_1(t), \ldots, \mathfrak{M}_N(t))$, the *conditioned second moment* whose elements are given by

$$\mathfrak{M}_i(t) = \mathbb{E}\big\{x(t)x'(t) \,|\, \eta(t) = i\big\}, \ \forall\, i = 1, \ldots, N..$$

Time reversed Markov jump linear systems

We assume an affine control law, given by $u(t) = K_{\theta(t)}(t)x(t) + k_{\theta(t)}(t)$. Then for each $i = 1, ldots, N$ we write

$$A_i(t) = Ai + B_i K_i(t) \quad \text{and} \quad b_i(t) = B_i k_i(t) + e_i(t).$$

The dynamic of the conditioned moment variables, are given by

$$\begin{aligned} \mathfrak{m}(t+1) &= \mathcal{E}\big(A(t)\mathfrak{m}(t) + b(t)\big), \\ \mathfrak{M}(t+1) &= \mathcal{U}\big(\mathfrak{M}(t)\big) + \mathcal{E}\big(A(t)\mathfrak{m}(t)b'(t) + b(t)\mathfrak{m}'(t)A'(t) + b(t)b'(t)\big), \end{aligned}$$

where

$$\mathcal{U}_{\mathbb{Z}}(\mathbb{Y}) = \sum_{j=1}^{N} p_{ij}(t) Z_j Y_j Z_j'$$

Time reversed Markov jump linear systems

Considering

$$y(t) = C_{\eta(t)}x(t) + D_{\eta(t)}u(t), \ \ \forall\, t = 0, 1, \ldots, \ell,$$

with $C_i D_i' = 0$ and $D_i D_i' > 0$ for each $i = 1, \ldots, N$.

The TRM-JLQ problem consist on: minimize the mean square of the variable y along the time ℓ, given by

$$J = \min_{u(0),\ldots,u(\ell)} \Big\{ \sum_{t=0}^{\ell} \mathbb{E}\big\{\|y(t)\|^2\big\} \Big\},$$

restricted to: $x(t+1) = A_{\eta(t)}(t)x(t) + b_{\eta(t)}(t)$ and $x(0) = x_0$.

Time reversed Markov jump linear systems

- Initialize with $\chi(\ell) = 0$. For each $i = 1, \dots, \ell$ compute

$$S_i(\ell) = \upsilon_i(\ell) C_i' C_i \quad \text{and} \quad s_i(\ell) = 0.$$

- For each $t = \ell - 1, \dots, 0$. If $\upsilon_i(t) = 0$ compute $S_i(t) = 0$ and $s_i(t) = 0$, otherwise

$$\begin{aligned}
\Sigma_i(t) &= \big(B_i' \mathcal{D}_i(S(t+1)) B_i + \upsilon_i(t) D_i' D_i\big)^{-1}, \\
\gamma_i(t) &= e_i'(t) \mathcal{D}_i(S(t+1)) + \frac{1}{2} \mathcal{D}_i'(s(t+1)), \\
S_i(t) &= \upsilon_i(t) C_i' C_i + A_i' \mathcal{D}_i(S(t+1)) A_i \\
&\quad - A_i' \mathcal{D}_i(S(t+1)) B_i \Sigma_i(t) B_i' \mathcal{D}_i(S(t+1)) A_i, \\
s_i(t) &= 2 A_i' \gamma_i'(t) - 2 A_i' \mathcal{D}_i(S(t+1)) B_i \Sigma_i(t) B_i' \gamma_i'(t).
\end{aligned}$$

Calculate

$$\chi(t) = \chi(t+1) + \sum_{\{i:\upsilon_i(t)>0\}} \Big\{ e_i'(t)\mathcal{D}_i(S(t+1))e_i(t) \\ + e_i'(t)\mathcal{D}_i(S(t+1))B_i\Sigma_i(t)B_i'\mathcal{D}_i(s(t+1)) - \gamma_i(t)\Sigma_i(t)\gamma_i'(t)\Big\}.$$

Also,

$$J = \langle \mathbb{S}(0), \mathfrak{M}(0)\rangle + s'(0)\mathfrak{m}(0) + \chi(0).$$

Ilustrative example

Consider a DC motor device with state $x(t) = (i(t), w(t))'$, where i is the motor current [A] and w is the angular velocity [rad/s]; $u(t)$ is the input voltage [V], and there is a nonlinear additive term and constants F_c and J.

$$\dot{x}(t) = \begin{bmatrix} -\frac{\mathrm{R}_m}{L} & -\frac{\mathrm{K}_e}{L} \\ \frac{\mathrm{K}_m}{J} & -\frac{\mathrm{K}_d}{J} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\frac{\mathrm{F}_c}{J} \end{bmatrix} \mathrm{sign}(w(t)).$$

Ilustrative example

Here we linearise and discretize with sampling time $T = 10^{-2}[s]$ and obtain

$$x(t+1) = A_{\mathrm{d}}x(t) + B_{\mathrm{d}}u(t) + e_{\mathrm{d}}, \;\; \forall\, t \in \mathbb{T}, \tag{1}$$

with

$$A_{\mathrm{d}} \approx \begin{bmatrix} -5.0027 \times 10^{-4} & -2.2887 \times 10^{-2} \\ 1.6495 \times 10^{-2} & 7.6453 \times 10^{-1} \end{bmatrix},$$

$$B_{\mathrm{d}} \approx \begin{bmatrix} 0.7906 \\ 7.6009 \end{bmatrix} \;\text{ and }\; e_{\mathrm{d}} \approx \begin{bmatrix} 0.8581 \\ -28.341 \end{bmatrix},$$

and assume that the motor s initialized with $x_0 = (2, -80)'$. We consider the controller memory acts naturally as a FILO queue, e.g. $(K(\ell), k(\ell))$ are the first gains to be stored and the last to be read and employed by the controller.

Ilustrative example

Assuming *Gilbert-Elliot channel* that may be in *high quality* or *low quality* state. These states are modeled by a hidden Markov chain with probability p_{hh} of permanence at high quality state and p_{ll} of permanence at low quality; when at "low", there is a chance p_{le} that there are errors in the transmission of a gain pair K, k, otherwise this chance is reduced to $p_{\text{he}} << p_{\text{le}}$. The four possible combinations (high,error), (high,no error), (low,error), (low,no error) are

Tabela: OPERATION MODES OF THE MOTOR

$\eta(t)$	Meaning of η
1	High quality channel / $K(t), k(t)$ contain errors
2	High quality channel / $K(t), k(t)$ contain no errors
3	Low quality channel / $K(t), k(t)$ contain errors
4	Low quality channel / $K(t), k(t)$ contain no errors

Ilustrative example

Assuming $p_{\mathrm{hh}} = 0.95$, $p_{\mathrm{ll}} = 0.99$, $p_{\mathrm{le}} = 0.9$, $p_{\mathrm{he}} = 0.001$ and $\upsilon(0) = (0, 1, 0, 0)'$. Regarding the information structure, in our basic implementation we assume that the controller has access to η and uses the stored gains when there is no error in $(K(t), k(t))$ otherwise sets $u(t) = 0$ (which is interpreted as "taking no action based on erroneous gains", recalling that $u = 0$ means applying the equilibrium voltage in the motor); this can be easily modelled by taking $B_2 = B_4 = B_{\mathrm{d}}$ and $B_1 = B_3 = 0$. As for the output process y we consider

$$C_i = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad \text{and} \quad D_i = 1, \text{ for each } i = 1, 2, 3, 4.$$

Ilustrative example

The simples controller is expected to have the worst performance in terms of cost, so that J_{ideal} is usually the highest cost. On the other extreme $J_{\text{optimistic}}$ is expected to be the smallest cost, since it corresponds to the case where there is no fault in the system.

Tabela: VARIABLE FOR EACH MODEL AND CONTROL

PARAMETER	MODEL	CONTROL
J_{optimal}	TRMJLS	OPTIMAL
J_{JLQR}	TRMJLS	CLASSIC
$J_{\text{optimistic}}$	TRMJLS	DETERMINISTIC
J_{ideal}	LS	DETERMINISTIC

Ilustrative example

Tabela: COST FUNCTIONS AND PERFORMANCE INDEXES

PERFORMANCE	$\eta_0 = 2$	$\eta_0 = 4$
$J_{\text{optimistic}}$	244.1697	375.5251
J_{JLQR}	243.8718	332.6154
J_{optimal}	243.6811	331.7862
J_{ideal}	242.4264	242.4264

Referencias

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THANK YOU.

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