A Multi-Dimensional Elephant Random Walk Moldel

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Acknowledgement:

Outline

Introduction → *Elephant Walk* → *Multi-Dimensional Generalization* → *Cow-Ox Model* → *Conclusions and Perspectives*

Introduction

- Elephant Random Walk was introduced by Schutz and Trimper in 2004
  - G.M. Schütz and S. Trimper, Phys. Rev. E **70**, 045101 (2004)
- Random Walk with Long-Range memory;
- Microscopic, one-dimensional model;
- Microscopic origin of memory;
- Analytical Solutions;

- I propose a multi-dimensional extension of the model;
- Coupling coefficients;
- Cow-Ox model: Particular two dimensional case with two newsworthy regimes

Introduction – Classical (biased) Random Walk, *an ordinary drunk*

$$X_{t+1} = X_t + \sigma_{t+1}$$

$$\begin{cases} P(\sigma_{t+1} = 1) = p \\ P(\sigma_{t+1} = -1) = 1 - p \end{cases} \Rightarrow \boxed{P(\sigma_{t+1} = \sigma) = \frac{1}{2}[1 + \sigma(2p - 1)]}$$

$$\boxed{X_t = X_0 + \sum_{i=1}^{t} \sigma_i}$$

Sum of independent, identically distributed random variables

$$\boxed{\langle X_t - X_0 \rangle = t(2p - 1)}$$

$$\boxed{\langle (X_t - X_0)^2 \rangle = 4tp(1 - p) = \sigma^2}$$

Linear on time

From central limit theorem, for $t \gg 1$:

$$\boxed{P(x_t = X_t - X_0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_t - \langle x_t \rangle)^2/2\sigma^2}}$$

Elephant Random Walk

$$X_{t+1} = X_t + \sigma_{t+1}$$

$$X_t = X_0 + \sum_{i=1}^{t} \sigma_i$$

- Probabilities of $\sigma_{t+1} = \sigma = \pm 1$?

Elephant Random Walk – *Game Rules*

- Rule 1: a time $t+1$, a number t' from $\{1, \dots, t\}$, is chosen with uniform probability $1/t$;
- Rule 2:

$$\sigma_{t+1} = \sigma_{t'} \text{, com probabilidade } p$$
$$\sigma_{t+1} = -\sigma_{t'} \text{, com probabilidade } 1-p$$

In other words:

$$P(\sigma_{t+1} = \sigma | \sigma_{t'}) = \frac{1}{2}[1 + \sigma_{t+1}\sigma_{t'}(2p-1)]$$

- Rule 3: at $t = 0$:

$$P(\sigma_1 = \sigma) = \frac{1}{2}[1 + \sigma(2q-1)]$$

Elephant Random Walk – *Results*

- $p = 1/2$: classical unbiased random walk;
- $p < 1/2$: *"dedicated (but not very stringent) reformer"**
- $p > 1/2$: *"is a more traditional type"**
- $p = 1$: Deterministic elephant

For the first and second displacement moment, for $t \gg 1$

$$\boxed{\langle x_t \rangle \sim \frac{\beta}{\Gamma(\alpha+1)} t^{\alpha}}$$

$$\boxed{\langle x_t^2 \rangle \sim \frac{t}{3-4p} \text{ para } p < 3/4}$$

$$\boxed{\langle x_t^2 \rangle \sim t\ln(t) \text{ para } p = 3/4}$$

$$\boxed{\langle x_t^2 \rangle \sim \frac{t^{4p-2}}{(4p-3)\Gamma(4p-2)} \text{ para } p > 3/4}$$

$$\alpha = 2p - 1$$
$$\beta = 2q - 1$$

*PRE **70**, 045101(R)(2004)

Multi-Dimensional Generalization

- *Problem: How to remember an N-dimensional step?*
- *Answer: Coupling N one-dimensional ERW!*

$$x_{t+1}^i = x_t^i + \sigma_{t+1}^i$$

Elephant Random Walk – *New Game Rules*

- Rule 1: at time $t+1$, N numbers t' from $\{1, \ldots, t\}$, are chosen with uniform probability $1/t$;
- Rule 2:

$$P\left(\sigma_{t+1}^i = \sigma \middle| \left\{\sigma_{t'_k}^k\right\}\right) = \sum_{k=1}^{N} \frac{1}{2}\left[1 + \sigma\sigma_{t'_k}^k \left(2p_k^i - 1\right)\right]\gamma_k^i$$

Coupling Coefficients

$$\sum_{k=1}^{d} \gamma_k^i = 1 \qquad 0 \le \gamma_k^i \le 1$$

- Rule 3: at $t = 0$:

$$P\left(\sigma_1^i = \sigma\right) = \sum_{k=1}^{N} \frac{1}{2}\left[1 + \sigma\left(2q_k^i - 1\right)\right]\gamma_k^i$$

Multi-Dimensional Generalization

$$\alpha_k^i = \left(2p_k^i - 1\right)$$
$$\beta_k^i = \left(2q_k^i - 1\right)$$

For the first Moments:

$$\langle x_{t+1}^i \rangle = \sum_{k=1}^{N} \left(\delta_{ki} + \frac{\gamma_k^i \alpha_k^i}{t} \right) \langle x_t^k \rangle \qquad \langle x_1^i \rangle = \sum_{k=1}^{N} \gamma_k^i \beta_k^i$$

For higher order moments the equations are harder. In general, they can be calculated by matrix recurrence equations:

$$\mathbb{M}_{t+1} = \mathbb{H}_t + \mathbb{G}_t \mathbb{M}_t$$

$$\mathbb{M}_t = \left(\prod_{k=t-1}^{1} \mathbb{G}_k \right) \mathbb{M}_1 + \sum_{i=1}^{t-2} \left(\prod_{k=t-1}^{i+1} \mathbb{G}_k \right) \mathbb{H}_i + \mathbb{H}_{t-1}$$

Cow-Ox Model

- “Bidimensional” model;
- Different interpretation: Not two different coupled dimensions, but two different coupled individuals walking on the same line;

- Direction 1 is the Cow;
- Direction 2 is the Ox;
- The Cow does not depend on the ox (it remembers only its own history);
- The Ox depends on the cow (it remembers both histories)

$$\text{Coupling } \Rightarrow \begin{cases} \gamma_1^1 = 1 \\ \gamma_2^1 = 0 \\ \gamma_1^2 = \gamma \neq 0 \\ \gamma_2^2 = 1 - \gamma \end{cases}$$

Cow-Ox Model

First Moment:

$$\begin{pmatrix}\langle x_{t+1}^1\rangle\\ \langle x_{t+1}^2\rangle\end{pmatrix}=\frac{1}{t}\begin{pmatrix}t+\alpha_1^1 & 0\\ \gamma\alpha_1^2 & t+(1-\gamma)\alpha_2^2\end{pmatrix}\begin{pmatrix}\langle x_t^1\rangle\\ \langle x_t^2\rangle\end{pmatrix}$$

$$\langle x_t^1\rangle\sim\frac{t^{\alpha_1^1}}{\Gamma(\alpha_1^1+1)}\langle x_1^2\rangle$$

$$\langle x_t^2\rangle\sim\frac{\gamma\alpha_1^2 t^{\boxed{\alpha_1^1}}}{(\alpha_1^1-(1-\gamma)\alpha_2^2)\Gamma(\alpha_1^1+1)}\langle x_1^1\rangle+t^{\boxed{(1-\gamma)\alpha_2^2}}\left[\frac{\langle x_1^2\rangle}{\Gamma((1-\gamma)\alpha_2^2+1)}+\gamma\alpha_1^2\langle x_1^1\rangle f(\gamma,\alpha_1^1,\alpha_2^2)\right]$$

Case of Interest:

$\alpha_1^1>(1-\gamma)\alpha_2^2$

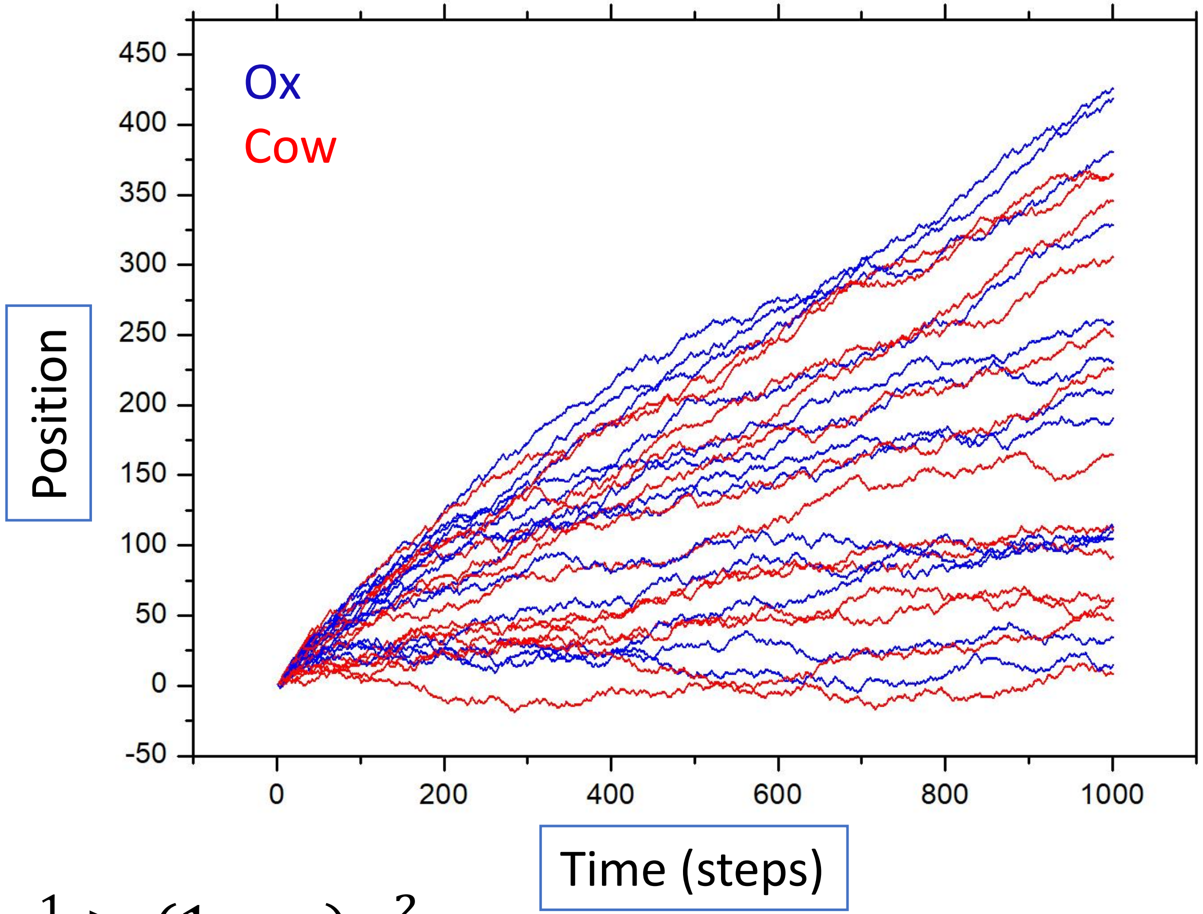
$$\boxed{\langle x_t^2\rangle\sim\frac{\gamma\alpha_1^2\langle x_t^1\rangle}{(\alpha_1^1-(1-\gamma)\alpha_2^2)}\begin{cases}\alpha_1^2>0\\ \alpha_1^2<0\end{cases}}$$

Attractive Regime

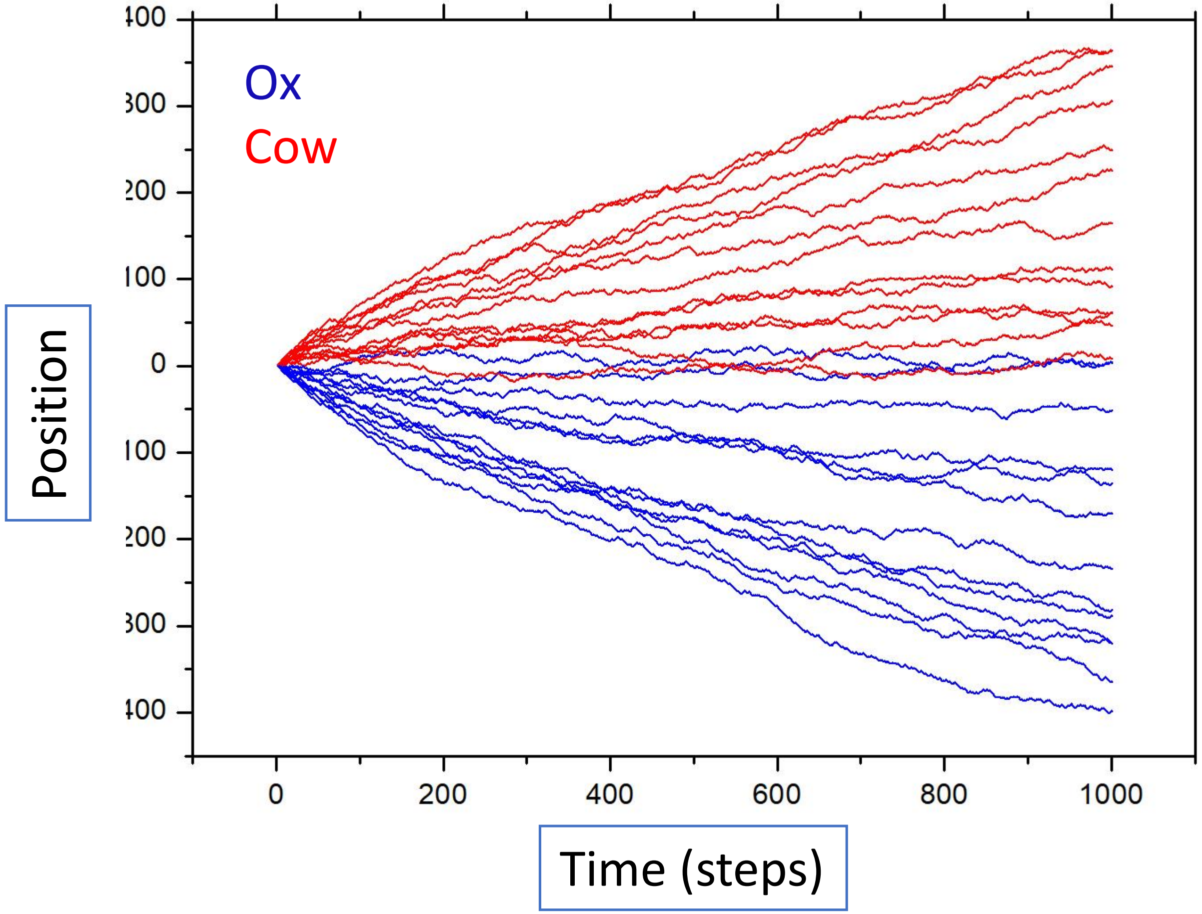
$\frac{\gamma\alpha_1^2}{(\alpha_1^1-(1-\gamma)\alpha_2^2)}<1$: Ox behind the Cow

Repulsive Regime

Attractive Regime $\alpha_1^2 > 0$



Repulsive Regime $\alpha_1^2 < 0$



$$\alpha_1^1 > (1-\gamma)\alpha_2^2$$

Cow-Ox Model

$$\boxed{x_{t+1}^i = x_t^i + \sigma_{t+1}^i} \times \boxed{x_{t+1}^j = x_t^j + \sigma_{t+1}^j}$$

Second Moment: $\mathbb{M}_{t+1} = \mathbb{H}_t + \mathbb{G}_t \mathbb{M}_t$

$$\mathbb{M}_t = \begin{pmatrix} \langle (x_t^1)^2 \rangle \\ \langle x_t^1 x_t^2 \rangle \\ \langle (x_t^2)^2 \rangle \end{pmatrix} \qquad \mathbb{H}_t = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbb{G}_t = \begin{pmatrix} 1 + 2\dfrac{\alpha_1^1 \gamma_1^1}{t} & 0 & 0 \\ \dfrac{\alpha_1^2 \gamma_1^2}{t} + \dfrac{\alpha_1^1 \gamma_1^1 \alpha_1^2 \gamma_1^2}{t^2} & \dfrac{(\alpha_1^1 \gamma_1^1 + \alpha_2^2 \gamma_2^2)}{t} + \dfrac{\alpha_1^1 \gamma_1^1 \alpha_2^2 \gamma_2^2}{t^2} & 0 \\ 0 & 2\dfrac{\alpha_1^2 \gamma_1^2}{t} & 1 + 2\dfrac{\alpha_2^2 \gamma_2^2}{t} \end{pmatrix}$$

$$\mathbb{M}_t = \left(\prod_{k=t-1}^{1} \mathbb{G}_k \right) \mathbb{M}_1 + \sum_{i=1}^{t-2} \left(\prod_{k=t-1}^{i+1} \mathbb{G}_k \right) \mathbb{H}_i + \mathbb{H}_{t-1}$$

Normal-diffusive regime

When $\alpha_1^2 = 0$:

$$\left\langle \left(x_t^i\right)^2 \right\rangle \sim \begin{cases} \dfrac{t}{1 - 2\alpha_i^i \gamma_i^i}, \alpha_i^i \gamma_i^i < 1/2 \\ t \ln t, \alpha_i^i \gamma_i^i = 1/2 \\ \dfrac{t^{2\alpha_i^i \gamma_i^i}}{\left(2\alpha_i^i \gamma_i^i - 1\right) \Gamma\left(2\alpha_i^i \gamma_i^i\right)}, \alpha_i^i \gamma_i^i > 1/2 \end{cases}$$

Super-diffusive regime

Cow-Ox M

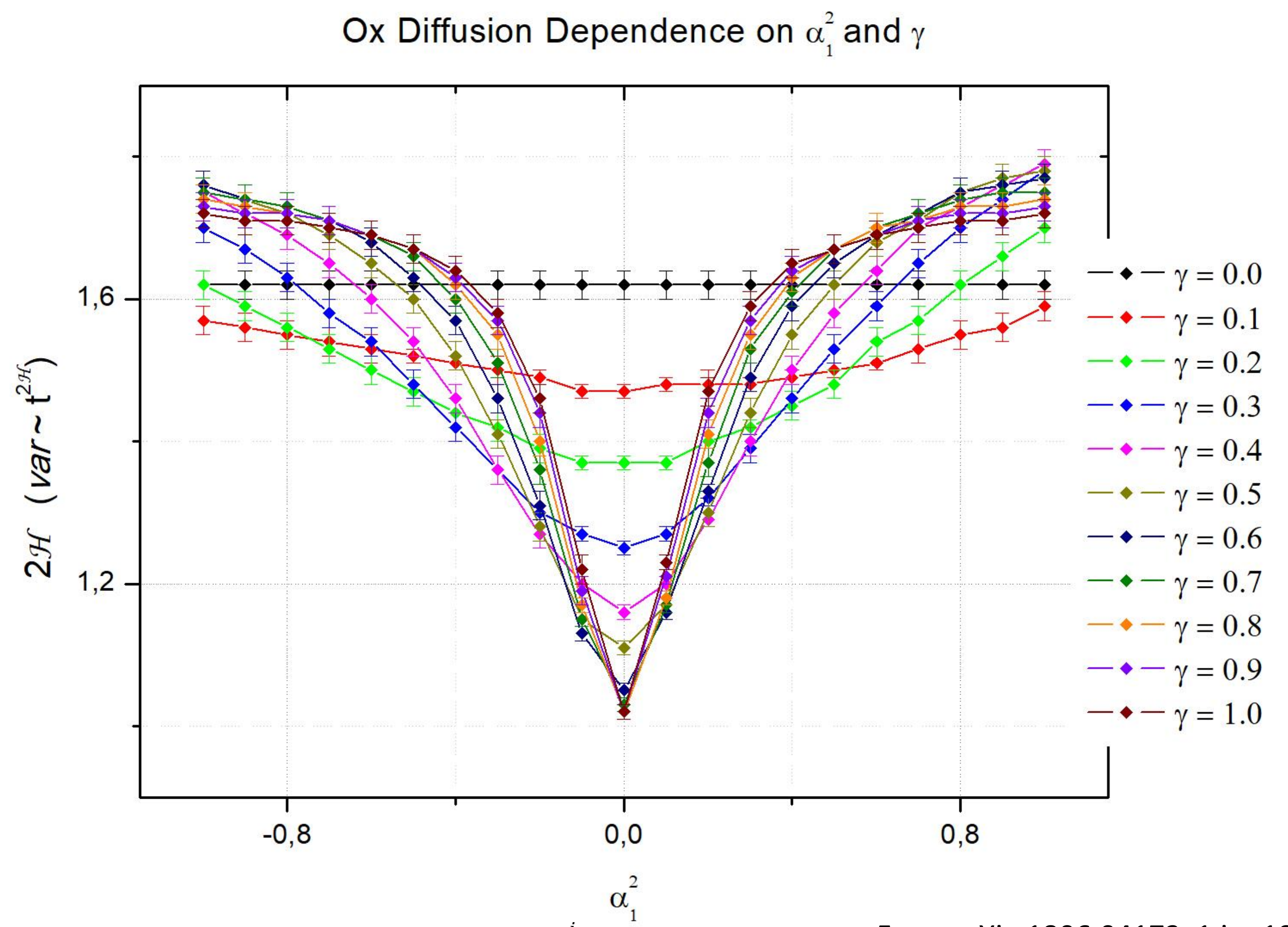
$$\ _{t+1}^{j} = x_t^j + \sigma_{t+1}^j$$

Second Mome

$$\mathbb{M}_t = \begin{pmatrix} \langle (x_t^1)^2 \rangle \\ \langle x_t^1 x_t^2 \rangle \\ \langle (x_t^2)^2 \rangle \end{pmatrix}$$

$$\mathbb{G}_t = \begin{pmatrix} 1 + 2 \\ \frac{\alpha_1^2 \gamma_1^2}{t} + \alpha \\ 0 \end{pmatrix}$$

$$\mathbb{M}_t = \left(\prod_{k=t-1}^{1} \mathbb{G}_k \right) \mathbb{M}_1 + \sum_{i=1}^{t-2}$$



10^3ox walking 10^3 steps each one; $\alpha_1^1 = \alpha_2^2 = 0.8$, $\alpha_2^1 = 0$ and $\beta_i^j = 1.0$

From arXiv:1806.04173v1 jun18

rmal-diffusive regime

$\overline{\gamma_i^i}, \alpha_i^i \gamma_i^i < 1/2$

$\alpha_i^i \gamma_i^i = 1/2$

$\overline{(2\alpha_i^i \gamma_i^i)}, \alpha_i^i \gamma_i^i > 1/2$

Super-diffusive regime

Jumping process – *general 2D case*

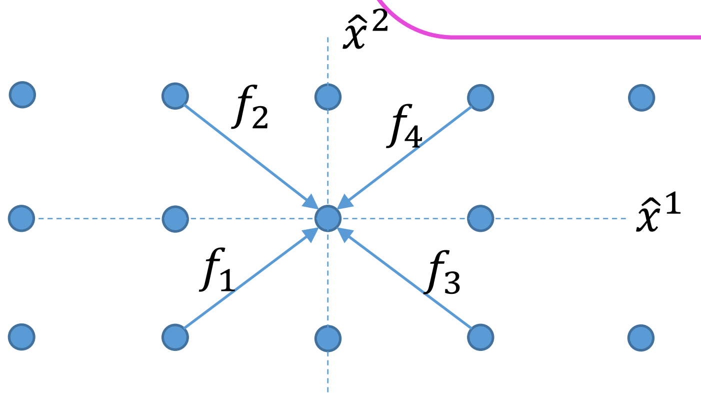
Characteristic function: $Q_t(\vec{k}) = \left\langle e^{i\vec{k}\cdot\vec{x}} \right\rangle$

$$
\begin{aligned}
Q_{t+1}(\vec{k}) = {} & \cos(k^1)\cos(k^2)\, Q_t(\vec{k}) \\
& + \cos(k^1)\sin(k^2) \sum_{i=1}^{2} \frac{\partial Q_t}{\partial k^i} \frac{\alpha_i^2 \gamma_i^2}{t} \\
& + \cos(k^2)\sin(k^1) \sum_{i=1}^{2} \frac{\partial Q_t}{\partial k^i} \frac{\alpha_i^1 \gamma_i^1}{t} \\
& + \sin(k^1)\sin(k^2) \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 Q_t}{\partial k^i \partial k^j} \frac{\alpha_i^1 \gamma_i^1 \alpha_j^2 \gamma_j^2}{t^2}
\end{aligned}
$$

Inverse Fourier Transform

Jumping Probabilities

$$
\begin{aligned}
P_{t+1}(x^1, x^2) = {} & P_t(x^1 - 1, x^2 - 1)\, f_1(\alpha_i^j \gamma_i^j, t, x^1 - 1, x^2 - 1) \\
& + P_t(x^1 - 1, x^2 + 1)\, f_2(\alpha_i^j \gamma_i^j, t, x^1 - 1, x^2 + 1) \\
& + P_t(x^1 + 1, x^2 - 1)\, f_3(\alpha_i^j \gamma_i^j, t, x^1 + 1, x^2 - 1) \\
& + P_t(x^1 + 1, x^2 + 1)\, f_4(\alpha_i^j \gamma_i^j, t, x^1 + 1, x^2 + 1)
\end{aligned}
$$



Conclusions & Perspectives

- Start: One-dimensional random walk – with long-range memory;
- Then: a multi-dimensional model with coupling coefficients;

- Cow-ox Model (particular bidimensional case);
- Two regimes: attractive and repulsive (when $\alpha_1^1 > (1-\gamma)\alpha_2^2$);
- Numerically calculated the variance;
- Jumping Process;
  - Time-inhomogeneous;
  - Position-dependent;

- I plan to study position-dependent coupling coefficients;
- I plan to introduce de cow-ox coupling in other types of memory random walks (ERW-like).

References

[1] G. M. Schütz and S. Trimper, Phys. Rev. E **70**, 045101(R) (2004)
[2] V. M. Marquioni arXiv:1806.04173v1 [cond-mat.stat-mech] 11Jun2018

P.s.: no animals were injured during the simulations.

Thank you!