The exclusion process with slow boundary

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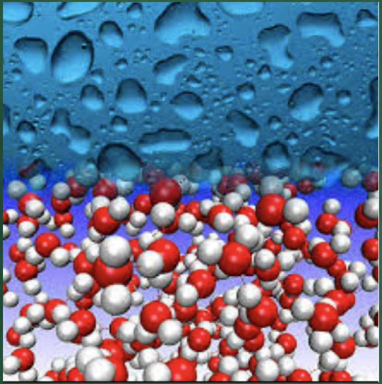
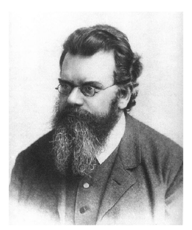
Interacting Particle Systems

Physical motivation

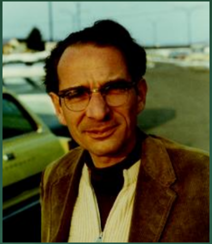


- Underlying scenario: we look for the evolution of a physical system, e.g. the spread of a gas confined to a finite volume.
- Two scales are considered:
  - a macroscopic one.
  - a microscopic one.
- Goal: describe the macroscopic evolution from the microscopic interaction between particles.
- Due to the huge number of molecules it is hard to describe precisely the microscopic state of a system.

History



Ludwig Eduard Boltzmann
(February 20, 1844 – September 5, 1906)

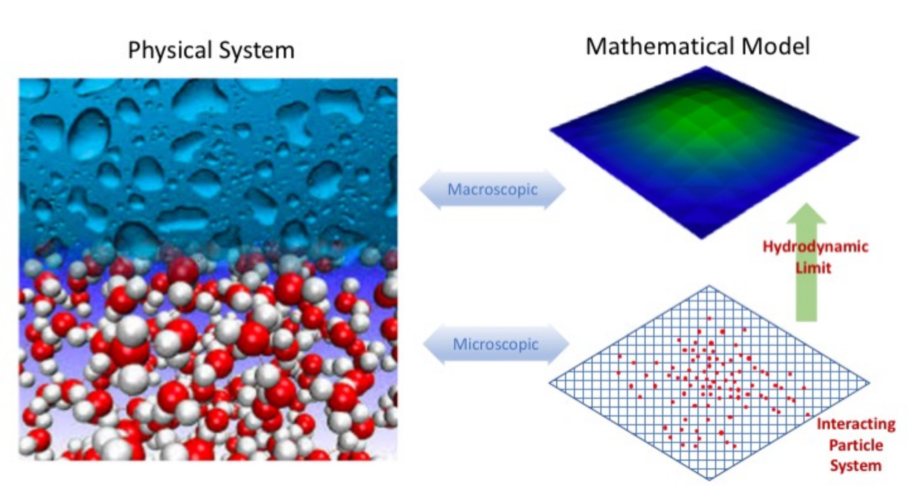


Frank Spitzer
(July 24, 1926 – February 1, 1992)

- Development of statistical mechanics.
- How the properties of atoms determine the physical properties of matter (such as viscosity, thermal conductivity, and diffusion)?

- Interaction of Markov processes. Advances in Mathematics, 1970.
- Each particle performs a *random walk* subject to some restriction.
- The random motion of particles is an interacting particle system.

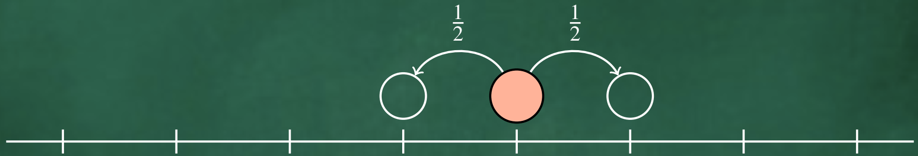
Scheme


Physical System
Mathematical Model
Macroscopic
Microscopic
Hydrodynamic
Limit
Interacting
Particle
System

Random Walk

Random walk

Consider a sequence of successive tosses of a fair coin and let $X_n = 1$ if the n-th toss is heads and $X_n = -1$ if the n-th toss is tails.



Let $X_0 = 0$ and for $n \geq 1$,
let $S_n = X_1 + \cdots + X_n$ be the position of the random walk at time n.
$\mathbb{E}[S_n] = 0$ and $Var(S_n) = \mathbb{E}[S_n^2] = n$, for any n.
For any $a, b \in \mathbb{R}$ with $a < b$:

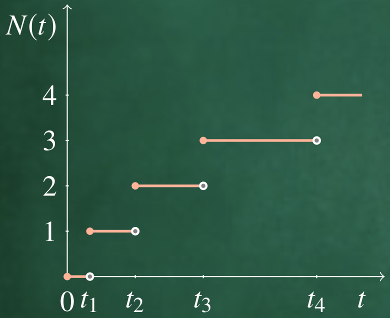
$$\lim_{n\to\infty} \mathbb{P}\left(a \leq \frac{S_n}{\sqrt{n}} \leq b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

Last statement is the **De Moivre-Laplace's** Central Limit Theorem.

Poisson Processes

Poisson Process I

Let us count the number of calls that arrive at a call center up to time t or the number of claims that arrive at an insurance company up to time t. Under certain hypothesis, each of these experiences can be represented as:



At the call center: the first call was received at time t_1, the second at time t_2 and so on.

At the insurance company: the first claim was received at time t_1, the second at time t_2 and so on.

Poisson Process II

Now let $A^k_{s,t}$ be the event "in the time interval $(s, s+t]$ the call center receives exactly k calls".
If we denote the previous result by $N(t)$ (note that it is a function of t), then the previous event corresponds to $N(s+t) - N(s) = k$, where N is a function belonging to the set of functions

$$\mathcal{X} := \{N : [0,\infty) \to \{0,1,2,\cdots\} : \exists t_1 < t_2 < \cdots$$
$$N(t) = 0, \text{for all } t \in [0,t_1), N(t) = 1, \text{for all } t \in [t_1,t_2), \cdots\}$$

Let us now see what are the hypothesis that we have to assume.

Poisson Process III

- Stationary increments: the probability that k calls arrive in $(s, s+t]$ depends only on t (the length of the interval): $\mathbb{P}(A^k_{s,t}) = \mathbb{P}(A^k_{0,t}) := P_k(t)$.
- Independent increments: the number of calls that arrive at disjoint time intervals are independent: $A^k_{s,t}$ is independent of $A^j_{u,v}$ for any choice of k and j as long as $(s, s+t] \cap (u, u+v] = \emptyset$.
- Calls do not arrive simultaneously: the conditional probability of having two or more calls during the time interval $(0, t]$ given that arrived one or more class in the same time interval, tends to 0 as $t \to 0$:

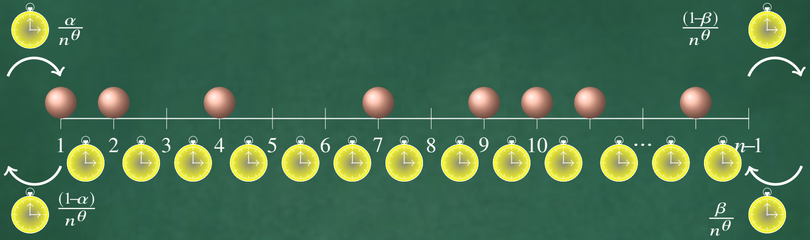
$$\frac{1 - P_0(t) - P_1(t)}{1 - P_0(t)} \to 0.$$

We conclude that $P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, where $\lambda = -\log(P_0(1))$.

Exclusion process with slow boundary

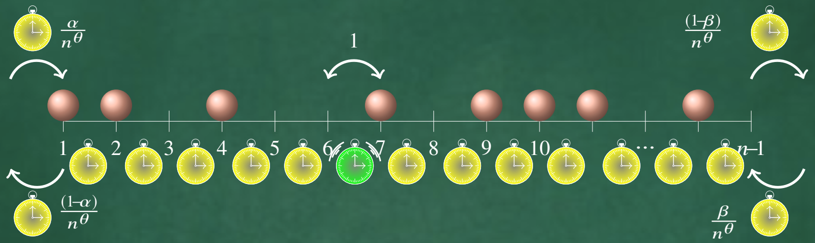
The exclusion process with slow boundary

The dynamics



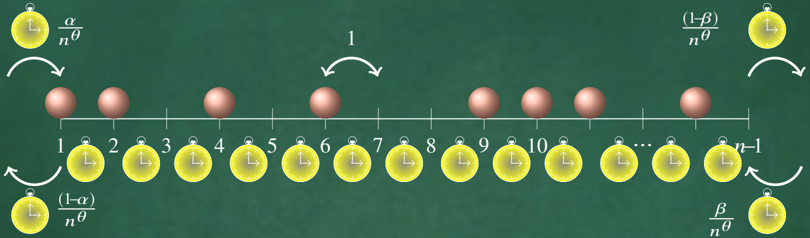
The exclusion process with slow boundary

The dynamics



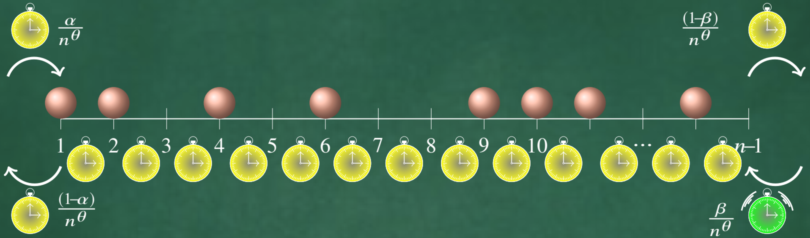
The exclusion process with slow boundary

The dynamics



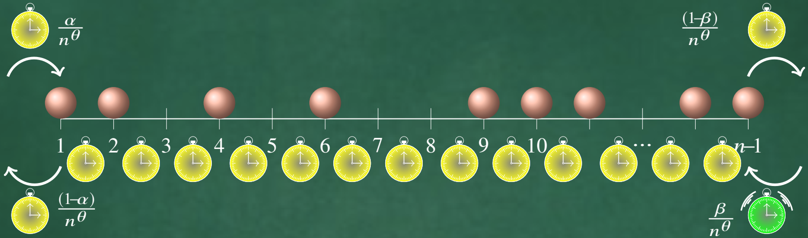
The exclusion process with slow boundary

The dynamics



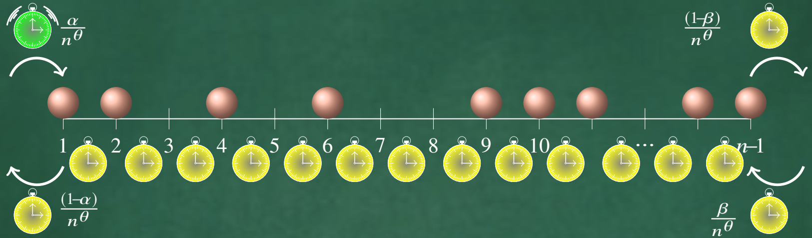
The exclusion process with slow boundary

The dynamics



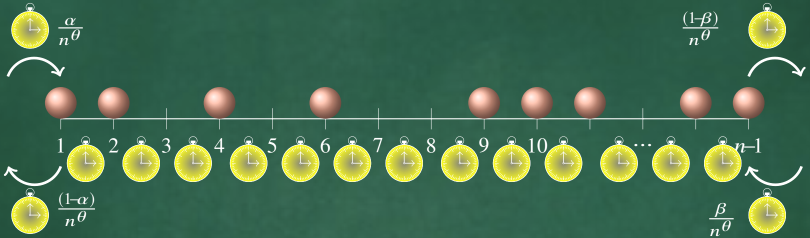
The exclusion process with slow boundary

The dynamics



The exclusion process with slow boundary

The dynamics



The dynamics - Infinitesimal Generator

Notations: $I_n = \{1, \dots, n-1\}$,

$$\eta(x) = \begin{cases} 1, & \text{if site x is occupied,} \\ 0, & \text{if site x is vacant.} \end{cases}$$

and $\eta = \{\eta(x)\} \in \{0,1\}^{I_n}$ is the configuration of particles.

$\{\eta_t = \{\eta_t(x)\},\ t \geq 0\}$ is the exclusion process with slow boundary.

The infinitesimal generator of this process is $L_n = L_{n,0} + L_{n,b}$, which acts on functions $f : \{0,1\}^{I_n} \to \mathbb{R}$ as

$$(L_{n,0} f)(\eta) = \sum_{x=1}^{n-2} \left[f(\eta^{x,x+1}) - f(\eta) \right]$$

and

$$\begin{aligned}(L_{n,b} f)(\eta) &= \left[\tfrac{\alpha}{n^\theta}(1-\eta(1)) + \tfrac{1-\alpha}{n^\theta}\eta(1) \right] \left[f(\eta^1) - f(\eta) \right] \\ &+ \left[\tfrac{\beta}{n^\theta}(1-\eta(n-1)) + \tfrac{1-\beta}{n^\theta}\eta(n-1) \right] \left[f(\eta^{n-1}) - f(\eta) \right],\end{aligned}$$

The spatial density of particles

Empirical measure

$$\tfrac{1}{n} I_n \subset [0,1]$$

$$\boxed{\pi^n(\eta, du) := \frac{1}{n} \sum_{x \in I_n} \eta(x)\, \delta_{\frac{x}{n}}(du)}$$

$$\int_0^1 H(u)\, \pi^n(\eta, du) = \frac{1}{n} \sum_{x \in I_n} \eta(x) H(\tfrac{x}{n}) \to \int_0^1 H(u)\, \rho(u)\, du$$

$$\pi^n(\eta, du) \Rightarrow \rho(u)\, du$$

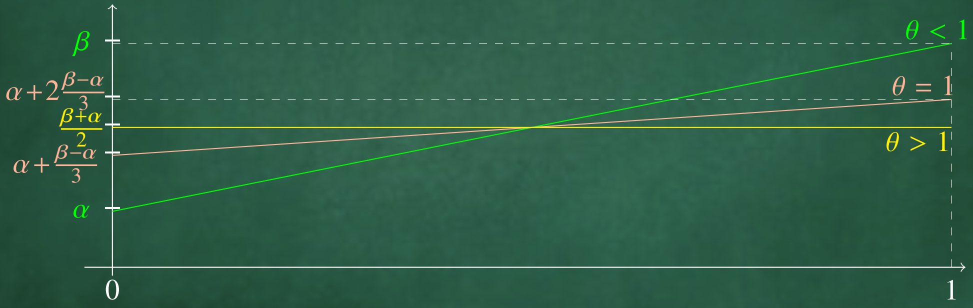
Hydrostatic Limit [Baldasso, Menezes, Neumann, Souza]

The stationary profile (density of particles)

Let μ_n^{ss} be the stationary probability measure in $\{0,1\}^{I_n}$ for the SSEP with slow boundary, which has infinitesimal generator given by $n^2 L_n$. Then, for all $\delta > 0$ and continuous function $H : [0,1] \to \mathbb{R}$,

$$\lim_{n\to+\infty} \mu_n^{ss}\Big[\Big|\frac{1}{n}\sum_{x\in I_n} \eta(x) H(\tfrac{x}{n}) - \int_0^1 H(u)\bar{\rho}(u)du\Big| > \delta\Big] = 0,$$

where $\bar{\rho} : [0,1] \to [0,1]$ is given by



Stationary probability μ_n^{ss}

- If $\alpha = \beta$, then the stationary (and also reversible) measure μ_n^{ss} is the Bernoulli product measure with parameter $\alpha = \beta \in (0,1)$,

$$\mu_n^{ss}(\eta) = \nu_\alpha^n(\eta) = \prod_{x \in I_n} [\eta(x)\alpha + (1-\eta(x))(1-\alpha)].$$

  In this case we say that the system is in equilibrium.
- If $\alpha \neq \beta$, then the stationary measure μ_n^{ss} is given by

$$\mu_n^{ss}(\eta) = \frac{1}{Z_{n-1}} \langle W| \prod_{x \in I_n} [\eta(x)D + (1-\eta(x))E]|V\rangle,$$

  where Z_{n-1} is a constant, D and E are square matrix (ansatz matrix) and $|V\rangle$ and $\langle W|$ are vectors satisfying:

$$\begin{aligned} DE - ED &= D + E \\ \langle W|\left(\tfrac{\alpha}{n^\theta}E - \tfrac{(1-\alpha)}{n^\theta}D\right) &= \langle W| \\ \left(\tfrac{\beta}{n^\theta}D - \tfrac{(1-\beta)}{n^\theta}E\right)|V\rangle &= |V\rangle. \end{aligned}$$

*B. Derrida, M. R. Evans, V. Hakin e V. Pasquier, 1993

Outline of the proof of hydrostatic limit

$$\mu_n^{ss}\left[\eta : \left|\frac{1}{n}\sum_{x\in I_n} H(\tfrac{x}{n})\,\eta(x) - \int_0^1 H(u)\,\overline{\rho}(u)\,du\right| > \delta\right]$$

- Compare $\bar{\rho}(\frac{x}{n})$ with $\rho^n(x) := \mathbb{E}_{\mu_n^{ss}}[\eta(x)]$ (the stationary empirical profile) in the following way:

$$\lim_{n\to\infty}\left(\max_{x\in I_n}\left|\rho^n(x) - \overline{\rho}(\tfrac{x}{n})\right|\right) = 0.$$

- Compare $\eta(x)$ with $\rho^n(x) := \mathbb{E}_{\mu_n^{ss}}[\eta(x)]$.

  To do this we need to bound the two points correlation function,

$$\varphi^n(x,y) := \mathbb{E}_{\mu_n^{ss}}\Big[\big(\eta(x) - \mathbb{E}_{\mu_n^{ss}}[\eta(x)]\big)\,\big(\eta(y) - \mathbb{E}_{\mu_n^{ss}}[\eta(y)]\big)\Big],$$

  in the following way

$$\max_{0<x<y<n}\left|\varphi^n(x,y)\right| \le \frac{C}{n^\theta + n},$$

  for some positive constant $C > 0$.

Empirical profile

Fix an initial measure μ_n in I_n. For $x \in I_n$ and $t > 0$, let

$$\rho_t^n(x) := \mathbb{E}_{\mu_n}[\eta_t(x)].$$

We extend the definition to the boundary by setting

$$\rho_t^n(0) = \alpha \text{ and } \rho_t^n(n) = \beta, \text{ for all } t \geq 0.$$

ρ_t^n is a solution of

$$\begin{cases} \partial_t \rho_t^n(x) = \Delta_n^\theta \rho_t^n(x), & t > 0, \quad x \in I_n, \\ \rho_t^n(0) = \alpha \ \text{ and } \ \rho_t^n(n) = \beta, & t > 0, \end{cases}$$

where Δ_n^θ acts on functions $f : I_n \cup \{0, n\} \to \mathbb{R}$ as

$$\begin{cases} \Delta_n^\theta f(x) = n^2(f(x+1) - f(x)) + n^2(f(x-1) - f(x)), \quad x \in \{2, \dots, n-2\}, \\ \Delta_n^\theta f(1) = n^2(f(2) - f(1)) + \frac{n^2}{n^\theta}(f(0) - f(1)), \\ \Delta_n^\theta f(n-1) = \frac{n^2}{n^\theta}(f(n) - f(n-1)) + n^2(f(n-2) - f(n-1)) \end{cases}$$

Stationary empirical profile

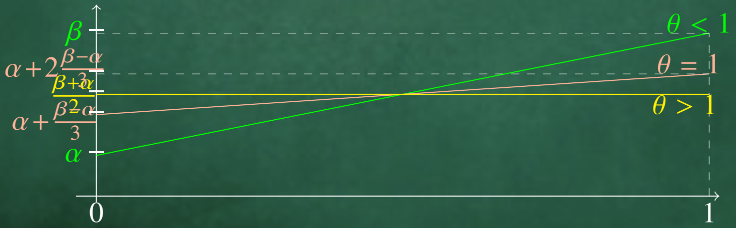
$\rho^n(x) = \mathbb{E}_{\mu_n^{ss}}[\eta(x)]$ is a solution of

$$\begin{cases} 0 = \Delta_n^\theta \rho^n(x) = n^2(\rho^n(x+1) - f(x)) + n^2(\rho^n(x-1) - \rho^n(x)), & x \in \{2, \dots, n-2\}, \\ 0 = \Delta_n^\theta \rho^n(1) = n^2(\rho^n(2) - \rho^n(1)) + \frac{n^2}{n^\theta}(\rho^n(0) - \rho^n(1)), \\ 0 = \Delta_n^\theta \rho^n(n-1) = \frac{n^2}{n^\theta}(\rho^n(n) - \rho^n(n-1)) + n^2(\rho^n(n-2) - f(n-1)). \end{cases}$$

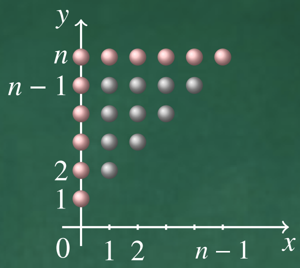
Then

$$\rho^n(x) = a_n\, x + b_n, \qquad x \in I_n,$$

where $a_n = \frac{\beta-\alpha}{2n^\theta+n-2}$ and $b_n = \alpha + a_n(n^\theta - 1)$.



The stationary correlation function of the occupation of two sites $x < y$



$\varphi^n(x,y)$ is a solution of

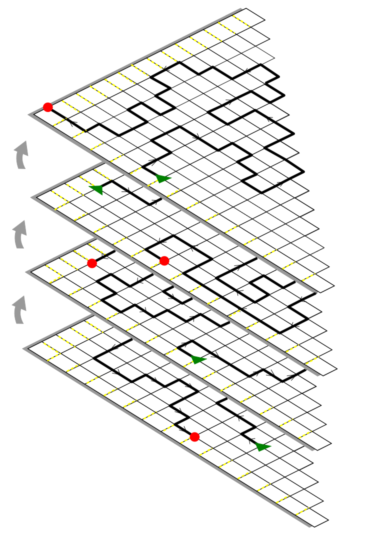
$$A_n^\theta \varphi^n(x,y) = a_n^2 \mathbf{1}_{y=x+1}, \text{ if } (x,y) \in V_n, \text{ and } \varphi^n(x,y) = 0 \text{ if } x = 0 \text{ or} y = n.$$

Then, is possible to compute that

$$\varphi^n(x,y) = -a_n^2 \cdot T_u^\theta,$$

where $T_u^\theta := \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\mathbf{X_u^\theta(s) \in D_n}}\, \mathbf{ds}\right]$ and $u = (x,y) \in V_n$.

If $\theta = 0$ and $u = (x, y)$, we have $T_u^0 = \frac{x(n-y)}{n-1}$.
We will compare T_u^0 and T_u^θ using a coupling.



$$\begin{aligned}
T_u^\theta &\le \mathbb{E}[D^{(1)} + \cdots + D^{(Y)}] \\
&= \sum_{j\ge 1} \mathbb{E}[(D^{(1)} + \cdots + D^{(j)})\mathbf{1}_{\mathbf{Y=j-1}}] \\
&= \sum_{j\ge 1} \mathbb{P}(Y = j-1) \sum_{i=1}^{j} T_{u_i}^0 \\
&\le \sum_{j\ge 1} \mathbb{P}(Y = j-1) C(n + j - 1) \\
&\le C(n + n^\theta).
\end{aligned}$$

$$|\varphi^n(x, y)| \le C(n + n^\theta) a_n^2.$$

Empirical profile

$\theta = 0$

$$\begin{cases} \partial_t \rho_t^n(x) = \Delta_n \rho_t^n(x), & t > 0, \quad x \in I_n, \\ \rho_t^n(0) = \alpha \ \text{ and } \ \rho_t^n(n) = \beta, & t > 0, \end{cases}$$

where Δ_n acts on functions $f : I_n \cup \{0, n\} \to \mathbb{R}$ as

$$\Delta_n f(x) = \Delta_n^0 f(x) = n^2(f(x+1) - f(x)) + n^2(f(x-1) - f(x)), \quad x \in I_n$$

$\theta = 1$ ($f(0) = \alpha$ and $f(n) = \beta$)

$$\begin{aligned} \Delta_n^1 f(1) &= n^2(f(2) - f(1)) + \frac{n^2}{n^1}(f(0) - f(1)) \\ &= n\big(\nabla_n^+ f(1) + (\alpha - f(1))\big) \end{aligned}$$

$$\begin{aligned} \Delta_n^1 f(n-1) &= \frac{n^2}{n^1}(f(n) - f(n-1)) + n^2(f(N-2) - f(n-1)) \\ &= n\big((\beta - f(n-1)) - \nabla_n^- f(n-1)\big) \end{aligned}$$

Hydrodynamic equation

$$\theta < 1: \begin{cases} \partial_t \rho(t,u) = \Delta\rho(t,u), & t > 0, \quad u \in (0,1), \\ \rho(t,0^+) = \alpha, & t > 0, \\ \rho(t,1^-) = \beta, & t > 0. \end{cases}$$

Dirichlet boundary conditions

$$\theta = 1: \begin{cases} \partial_t \rho(t,u) = \Delta\rho(t,u), & t > 0, \quad u \in (0,1), \\ \partial_u \rho(t,0^+) = \rho(t,0^+) - \alpha, & t > 0 \\ \partial_u \rho(t,1^-) = \beta - \rho(t,1^-), & t > 0. \end{cases}$$

Robin boundary conditions

$$\theta > 1: \begin{cases} \partial_t \rho(t,u) = \Delta\rho(t,u), & t > 0, \quad u \in (0,1) \\ \partial_u \rho(t,0^+) = 0, & t > 0, \\ \partial_u \rho(t,1^-) = 0, & t > 0. \end{cases}$$

Neumann boundary conditions

Empirical profile

$\theta > 1$

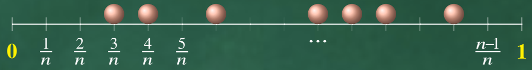
$$\begin{cases} \Delta_n^\theta f(1) = n\nabla_n^+ f(1) - \frac{n}{n^\theta}\nabla_n^+ f(0) \\ \Delta_n^\theta f(n-1) = \frac{n}{n^\theta}\nabla_n^+ f(n-1) - n\nabla_n^- f(n-1) \end{cases}$$

$\theta \in (0,1)$ ($f(0) = \alpha$ and $f(n) = \beta$)

$$\begin{cases} \Delta_n^\theta f(1) = n^2(f(2) - 2f(1) + f(0)) + n^2\left(1 - \frac{1}{n^\theta}\right)(f(1) - \alpha) \\ \Delta_n^\theta f(n-1) = n^2\left(1 - \frac{1}{n^\theta}\right)(f(n-1) - \beta) + n^2(f(N-2) - 2f(n-1) + f(n)) \end{cases}$$

Time evolution of the spatial density of particles

Empirical measure



$$\pi_t^n(du) \;=\; \frac{1}{n}\sum_{x\in I_n} \eta_{tn^2}(x)\,\delta_{\frac{x}{n}}(du)$$

$$\pi_t^n \in \mathcal{M} := \{\mu \text{ positive measure on } [0,1];\ \mu([0,1]) \leq 1\}$$

Fix $T > 0$.

- $\{\pi_t^n; 0 \leq t \leq T\}$ belongs to $D([0,T],\mathcal{M})$;
- $\{\pi_t^n; 0 \leq t \leq T\}$ inherits the Markov property from $\{\eta_t; 0 \leq t \leq T\}$;

Hydrodynamic Limit

The time evolution of the spacial density of particles is given by

$$\pi_t^n(du) \;=\; \frac{1}{n}\sum_{x=1}^{n-1} \eta_{tn^2}(x)\,\delta_{\frac{x}{n}}(du).$$

Theorem [Baldasso, Menezes, Neumann, Souza]

Let $\gamma : [0,1] \to [0,1]$ be a profile and $\{\mu_n\}_n$ a sequence of initial distributions such that

$$\pi_0^n(du) \;\Rightarrow\; \gamma(u)\,du.$$

Then, for all $t \geq 0$, we have

$$\pi_t^n(du) \;\Rightarrow\; \rho(t,u)\,du,$$

where ρ is the unique weak solution of

$$\partial_t \rho(t,u) = \partial_u^2 \rho(t,u),$$

with initial condition $\rho(0,u) = \gamma(u)$ and the boundary conditions depends on θ.

ρ is the unique weak solution of

$$\partial_t \rho(t,u) = \partial_u^2 \rho(t,u),$$

with initial condition $\rho(0,u) = \gamma(u)$ and the boundary conditions depends on θ:

- $\theta < 1$: $\rho(t,0) = \alpha$, $\rho(t,1) = \beta$;
- $\theta = 1$: $\partial_u \rho(t,0) = \rho(t,0) - \alpha$, $\partial_u \rho(t,1) = \beta - \rho(t,1)$;
- $\theta > 1$: $\partial_u \rho(t,0) = 0$, $\partial_u \rho(t,1) = 0$.

Thank you!