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Lecture Notes
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AN INTRODUCTION TO FIRST PASSAGE PERCOLATION

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Contents

1. Introduction. Motivation.

First passage percolation (fpp) refers to a probabilistic model introduced by Hammersley and Welsh in 1965 (see [16]) to simulate the flow of a fluid through random medium. It is simple to describe and at the same time poses big challenges, with plenty of interesting questions. After important developments obtained in the 80s and 90s, the topic has progressed considerably in the last decade. Yet many basic questions remain to be settled. In [5] the reader can find an excellent recent survey, which besides the discussion of late developments, contains proofs of the classical results. The aim of these four lectures is to provide a small introduction to the subject. The time limits substantially the material that can be covered. At the end we shall give further indications, and the students who are willing to get into more details after these lectures are directed to [20] and [5].

The basic model consists in assigning non-negative random variables to the edges of a graph; they are usually thought to be i.i.d. and to represent the time for the fluid to cross the given edge. One of the first natural questions deals with the description of the set of vertices reached by the flow at a given large time, assuming that it starts at a given fixed vertex. Given two vertices x, y one may also inquire about the fastest way to go from x to y. This defines a random pseudo-metric (a true metric if the passage times are strictly positive) and it brings immediately many interesting and hard questions.

Let us consider the most classical situation: the graph is the usual cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ i.e. the vertices are represented by the points in the d-dimensional lattice with integer coordinates, $\mathbb{Z}^d$, with edges $e =< x, y >$ joining two vertices $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ if and only if $|x - y| := \sum_i |x_i - y_i| = 1$. We consider independent and identically distributed random variables $\{\tau(e) \colon e \in \mathbb{E}^d\}$. Let F denote their distribution and we assume $F(0-) = 0$. Further assumptions will appear later. A path $\pi = (e_1, \dots, e_k)$ is a sequence of adjacent edges (sharing a vertex), i.e. $e_i =< x_{i-1}, x_i >$ for each $i = 1, \dots, k$, In this case we say that π goes from x_0 to x_k. We similarly define an infinite path $\pi = (e_1, e_2, \dots)$. The travel time of a finite path π is defined as

$$t(\pi) = \sum_{e \in \pi} \tau(e). \tag{1.1}$$

For $x, y \in \mathbb{Z}^d$ distinct we set

$$t(x, y) = \inf\{t(\pi) \colon \pi \text{ goes from } x \text{ to } y\} \tag{1.2}$$

and for sake of definition we let $t(x, x) = 0$. More generally, if $A, B \subset \mathbb{Z}^d$

$$t(A, B) = \inf\{t(\pi) \colon \pi \text{ goes from } A \text{ to } B\}. \tag{1.3}$$

Remark. Since the model becomes rather trivial in the one-dimensional case, we take $d \geq 2$ throughout.

The following are basic objects to be understood:
(**Notation:** $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^d$.)

$$\tilde{B}(t) = \{x \in \mathbb{Z}^d \colon t(\mathbf{0}, x) \leq t\}, \tag{1.4}$$

which consists of all the vertices that are reached from the origin up to time t. For convenience in some statements, one replaces $\tilde{B}(t)$ by

$$B(t) = \tilde{B}(t) + [-1/2, 1/2)^d, \tag{1.5}$$

where $A + C = \{x + y \colon x \in A, y \in C\}$, for A, C are non-empty subsets of $\mathbb{R}^d$.

The first two basic questions regard:

- The asymptotic behaviour of $t(\mathbf{0}, x)$ as $|x| \to +\infty$.
- The asymptotic shape of $B(t)$ as $t \to +\infty$.

One of the most important classical results treating these questions can be stated as follows.

Theorem 1.1. (Cox-Durrett shape theorem [9].**)** *Assume the random variables $\tau(e), e \in \mathbb{E}^d$ to be i.i.d. with common distribution F such that $F(0-) = 0$ and such that*

$$E\left(\min\{t_1^d, \dots, t_{2d}^d\}\right) < \infty, \tag{1.6}$$

for $t_1, \dots, t_{2d}$ i.i.d. with distribution F. Then, there exists a convex set $B_0 \subset \mathbb{R}^d$ which has a non-empty interior and is either compact or coincides with $\mathbb{R}^d$, such that the following holds:
a) If B_0 is compact, then for all $\epsilon > 0$

$$P\left((1-\epsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\epsilon)B_0, \text{ for all large } t\right) = 1. \tag{1.7}$$

b) If $B_0 = \mathbb{R}^d$, then for all K compact in $\mathbb{R}^d$

$$P\left(K \subset \frac{1}{t}B(t) \text{ for all large } t\right) = 1. \tag{1.8}$$

The set B_0 is invariant under permutations of the coordinates or reflections in the coordinate hyperplanes.

When (1.6) fails one has

$$\limsup_{|v|\to\infty} \frac{1}{|v|} t(0,v) = \infty \qquad a.s. \tag{1.9}$$

The first asymptotic shape theorem for $B(t)$ was obtained by Richardson in the seminal paper [29].

After the previous theorem it is natural to ask for the conditions under which B_0 is compact which means that $B(t)$ grows with a linear speed. This question was answered by Kesten.

Theorem 1.2. (Kesten [17].) *Under the conditions of Theorem 1.1, $B_0 = \mathbb{R}^d$ if and only if $F(0) \geq p_c(\mathbb{Z}^d)$, the critical percolation for Bernoulli bond percolation in $\mathbb{Z}^d$.*

Before moving to the proof of Theorem 1.1 we proceed, following as well the historical development, to the study of the asymptotic behavior of $\tilde{B}(t)$'s extreme points along the coordinate axes.

Notation: $\mathbf{e}_i, 1 \leq i \leq d$ denote the canonical basis in $\mathbb{R}^d$: $\mathbf{e}_1 = (1, 0, \dots, 0)$, etc.

Theorem 1.3. *If*

$$E(\min\{t_1, \dots, t_{2d}\}) < \infty, \tag{1.10}$$

where $t_1, \dots, t_{2d}$ are i.i.d. with distribution F, then there exists a finite constant $\mu \in [0, \infty)$ so that

$$\lim_{n\to\infty} \frac{1}{n} t(0, n\mathbf{e}_1) = \mu \quad a.s. \text{ and in } L^1. \tag{1.11}$$

Before proceeding with the proofs of the basic results that form the core of this introductory course, it is proper to say a few words on a famous precursor of this model. In 1961, to study the growth of a colony of cells constrained to a surface, Eden [14] introduced the following model in $\mathbb{Z}^2$: start with a cell at the origin: $A_0 = \{\mathbf{0}\}$. At the next step, add one of the four neighbors with equal probability, so that A_1 is a set of two neighbor vertices (or sites). A_1 has six neighbor sites and A_2 is obtained from A_1 by the addition of one of these, picked at random with equal probabilities. We continue this process: A_{n+1} is obtained from A_n by the addition of a single vertex, chosen uniformly among the neighbors of A_n. Of course, this construction can be generalized to any dimension. In [29], Richardson related this to the site first passage percolation in $\mathbb{Z}^d$ with i.i.d. exponential passage times attached to the vertices, in the sense that if we let $\tau_0 = 0$, $\tau_n = \inf\{t > 0\colon \tilde{B}(t)$ has $n+1$ vertices $\}$, where $\tilde{B}(t)$ is defined as in (1.4) for this site model, then the process $(A_n\colon n \ge 1)$ and $(\tilde{B}(\tau_n)\colon n \ge 1)$ have the same distribution.

These lectures provide a small introduction, focused on the most basic results. There is a vigorous development of the subject in the last years as seen if one takes a look at the survey article [5] and some of the recent research articles quoted therein. On the other hand, it is also important to say that many basic questions remain unanswered, showing their difficulty. For this, it suffices to glance the list of open problems in [5].

The general plan for the lectures is the following: We start by recalling the sub-additive ergodic theorem (without giving the proof), which is a basic tool in the subject. With this tool in hands, one can easily prove Theorem 1.3. We then proceed to the proof of Theorem 1.1, essentially following [9] and [20]. In the following lectures we discuss some aspects of the comparison of time constants and some related problems. Some further examples are commented at the end.

2. Preliminaries. Sub-additive ergodic theorem.

Subadditivity is an essential tool in first passage percolation. As we shall see, Theorem 1.3 can be obtained by an application of a sub-additive ergodic theorem. Here, we state Liggett's subadditive ergodic theorem ([25], Theorem VI.2.6), which improves Kingman's earlier version [21, 22] and is well adapted to the purposes of these notes.

Theorem 2.1. **(Sub-additive Ergodic Theorem.)** *Let us suppose that* $(X_{m,n}\colon 0 \le m \le n)$ *are random variables satisfying the following properties:*
(i) $X_{0,0} = 0, X_{0,n} \le X_{0,m} + X_{m,n}$ *for* $m \le n$.
(ii) For each $k \ge 1$, *the sequence* $(X_{(n-1)k,nk}\colon n \ge 1)$ *is stationary.*
(iii) For each $m \ge 0$, *the sequences* $(X_{m,m+k}\colon k \ge 0)$ *and* $(X_{m+1,m+k+1}\colon k \ge 0)$ *have the same distribution.*
(iv) $EX^+_{0,1} < \infty$. *Let* $\alpha_n = EX_{0,n}$, *which is well defined under (i), (ii) and (iv). Then*

$$\alpha = \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf_{n\ge 1} \frac{\alpha_n}{n} \in [-\infty, \infty), \tag{2.1}$$

and

$$X_\infty := \lim_{n\to\infty} \frac{X_{0,n}}{n} \text{ exists a.s., with } -\infty \le X_\infty < \infty. \tag{2.2}$$

Furthermore $EX_\infty = \alpha$. *If* $\alpha > -\infty$, *then* $\frac{X_{0,n}}{n}$ *converges to* X_∞ *in* L^1. *If the stationary processes in (ii) are ergodic,* $X_\infty = \alpha$ *a.s.*

3. The time constant. Proof of Theorem 1.3

The proof of Theorem 1.3 is obtained by applying Theorem 2.1 to the variables $X_{m,n} = t(m\mathbf{e}_1, n\mathbf{e}_1)$. For this we need to verify conditions (i) to (iv). Condition (i) follows at once from the definition: the concatenation of a path from $\mathbf{0}$ to $m\mathbf{e}_1$ with a path from $m\mathbf{e}_1$ to $n\mathbf{e}_1$ yields a path from $\mathbf{0}$ to $n\mathbf{e}_1$, and it follows that

$$t(\mathbf{0}, n\mathbf{e}_1) \le t(\mathbf{0}, m\mathbf{e}_1) + t(m\mathbf{e}_1, n\mathbf{e}_1)$$

(Indeed, by the same reason we see that $t(x,y) \le t(x,z) + t(z,y)$ for any $x, y, z \in \mathbb{Z}^d$, i.e. $t(\cdot,\cdot)$ satisfies the triangular inequality.) Conditions (ii) and (iii) and the ergodicity of the family follow at once from the fact that the $\tau(e), e \in \mathbb{E}^d$ are i.i.d. It remains to show that $Et(\mathbf{0}, \mathbf{e}_1) < \infty$. For this one uses condition 1.10 and the existence of $2d$ disjoint paths $\pi_1, \ldots, \pi_{2d}$ from $\mathbf{0}$ to $\mathbf{e}_1$ (See Lemma 4.1). Thus

$$t(\mathbf{0}, \mathbf{e}_1) \le \min\{t(\pi_1), \ldots, t(\pi_{2d})\},$$

And therefore

$$\begin{aligned} P(t(\mathbf{0}, \mathbf{e}_1) > s) \le \prod_{i=1}^{2d} P(t(\pi_i) > s)) &\le K^{2d}(P(\tau(e) > s/K))^{2d} \\ &= K^{2d} P(Z > s/K), \end{aligned}$$

where K is an upper bound for the number of edges in all these $2d$ paths and let $Z = \min\{t_1, \ldots, t_{2d}\}$ as in the statement. We just used that if a path π has at most K edges, then $P(t(\pi_i) > s) \le KP(\tau(e) > s/K)$. $\square$

An immediate question is: what does one know about the time constant μ? Of course $0 \le \mu \le E(\tau(e))$. The question as to whether $\mu = 0$ or $\mu > 0$ has been answered by Kesten. Indeed, from the discussion in the next section, it is easy to see that $\mu = 0$ in Theorem 1.3 if and only if $B_0 = \mathbb{R}^d$ in Theorem 1.2.

Exercise. If the distribution F is not degenerate, then $\mu < E(\tau(e))$.

Remark 3.1. *(i) As seen from the proof, under condition* (1.10)*, the time constant* μ *in Theorem 1.3 is given by*

$$\mu = \inf_{n \ge 1} \frac{1}{n} E(t(0, n\mathbf{e}_1)).$$

(ii) Let us assume that condition (1.10) *does not hold. Note that*

$$t(\mathbf{0}, n\mathbf{e}_1) \ge \min\{\tau(f_{n,1}), \ldots \tau(f_{n,2d})\}$$

where $f_{n,1}, \ldots, f_{n,2d}$ *are the edges incident to* $n\mathbf{e}_1$. *These set of edges are pairwise disjoint when* n *takes even values. It then follows by a simple* *application of Borel-Cantelli that* $P(t(\mathbf{0}, 2n\mathbf{e}_1) \geq an \ i.o\ ) = 1$ *for any* $a > 0$, *which implies at once that*

$$\limsup_{n\to\infty} \frac{1}{n} t(\mathbf{0}, n\mathbf{e}_1) = \infty \text{ a.s.} \tag{3.1}$$

(iii) One should also remark that no moment condition is needed in order guarantee the existence of a constant μ *such that*

$$\lim_{n\to\infty} \frac{1}{n} t(\mathbf{0}, n\mathbf{e}_1) = \mu \textit{ in probability.} \tag{3.2}$$

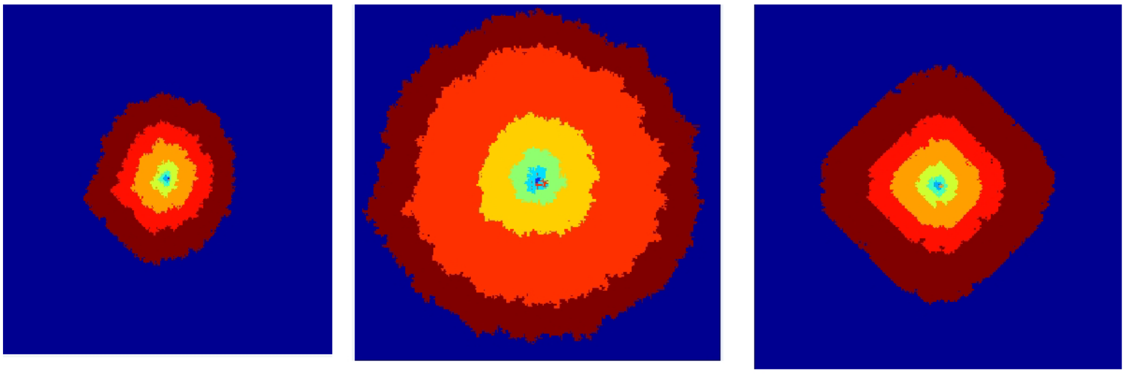
This last result was first proved by Wierman in [32], and then extended by Cox-Durrett in [9] when $d = 2$, and by Kesten (see [17]) for $d \geq 2$. The basic idea (see [30] and [9]) is to replace $t(x, y)$ by another random variable $\hat{t}(x, y)$ which denotes the minimum time between suitable (random) neighborhoods of x and y respectively. This is done in such a way that: (a) the random variables $\hat{t}(x, y)$ will satisfy the needed integrability conditions and be close to a subbaditive system so that $\hat{t}(\mathbf{0}, n\mathbf{e}_1)/n \to \mu$ a.s., and (b) $(t(\mathbf{0}, n\mathbf{e}_1) - \hat{t}(\mathbf{0}, n\mathbf{e}_1))/n$ will tend to zero in probability. The construction of these neighborhoods is simpler when $d = 2$, where it goes as follows:

Fix M so that $F(M) > 1/2$, and declare each edge e to be open if $\tau(e) \leq M$, closed otherwise. The critical parameter for Bernoulli edge percolation on $\mathbb{Z}^2$ is $1/2$. For this choice of M there is a unique infinite cluster of open edges and no infinite cluster of closed edges. This implies that for each $z \in \mathbb{Z}^2$ we may take a minimal open circuit $\Delta(z)$ [1] which contains x in its interior and is part of the infinite open cluster. One may define $\bar{\Delta}(z)$ as the set that consists of $\Delta(z)$ and all the edges inside the circuit. The new time variables are $\hat{t}(x, y) = t(\bar{\Delta}(x), \bar{\Delta}(y))$, as in (1.3). Using this, it is not hard to see that

$$\hat{t}(x, y) \leq t(x, y) \leq \hat{t}(x, y) + u(x) + u(y)$$

where $u(z) = \sum_{e\in\bar{\Delta}(z)} \tau(e)$, which has a distribution that does not depend on z. This proof gets more technical. We refer to [9] for the two dimensional case or [17] for $d \geq 2$. The argument also gives that $\mu = \liminf_{n\to\infty} \frac{1}{n} t(\mathbf{0}, n\mathbf{e}_1)$ almost surely.

[1] a path $x_0, x_1, \ldots, x_{k-1}, x_k$ where all the edges $< x_i, x_{i+1} >$ are distinct as well $x_0, \ldots, x_{k-1}$ and $x_0 = x_k$.



Some illustrations (simulations by Graccyela Salcedo)

4. Cox-Durrett shape theorem

The basic idea of the proof of Theorem 1.1 is to start by using subadditivity to get the (linear) growth of $B(t)$ in each rational direction. This is quite similar to what was done in the proof of Theorem 1.3. It yields the correct growth rate in all rational directions simultaneously, with probability one. To obtain the full result one needs an estimate that allows to control the difference of the rates along nearby directions, so as to interpolate between them and get the required a.s. behaviour in all directions.

We essentially follow [9] with trivial adaptations. The proof is slightly simpler under the stronger moment condition $\int x^2 dF(x) < \infty$. (see [20].)

Exercise. Show that if the distribution F has a finite first moment, then condition (1.6) holds.

We start with a technical lemma concerning disjoint paths joining the same pair of points.

Lemma 4.1. *Let x and y be distinct points of $\mathbb{Z}^d$ with $d \geq 2$. Then, there exists $2d$ disjoint paths of length at most $|x-y|+6$ from x to y*

Proof. Without loss of generality, we assume that $x = 0$ and we proceed by induction on d. For $d = 2$ we analyze two cases:
a) One of the $y = (y_1, y_2)$ coordinates is 0. Again without loss of generality we assume that $y_2 = 0$ and that $y_1 = n > 0$. Then $|x - y| = n$ and we exhibit 4 disjoint paths of length at most $n + 6$ giving the successive points they visit:
$(0,0), (0,1), \ldots, (0,n)$
$(0,0), (0,1), (1,1), \ldots, (n,1), (n,0)$
$(0,0), (-1,0), (-1,-1), (0,-1), (1,-1), \ldots, (n+1,-1), (n+1,0), (n,0)$
$(0,0), (0,-1), (0,-2), (1,-2), \ldots, (n,-2), (n,-1), (n,0)$
b) Both y_1 and y_2 are not 0. Again without loss of generality we assume that $y_1 = n > 0$ and $y_2 = m > 0$ and give the 4 disjoint paths of length at most $n + m + 4$.

$(0,0),(0,1),\ldots,(0,m),(1,m),\ldots,(n,m)$
$(0,0),(1,0),\ldots,(n,0),(n,1),\ldots,(n,m)$
$(0,0),(-1,0),(-1,1),\ldots,(-1,m{+}1),(0,m{+}1),(1,m{+}1),\ldots(n,m{+}1),(n,m)$
$(0,0),(0,-1),(1,-1),\ldots,(n+1,-1),(n+1,0),\ldots,(n+1,m),(n,m)$
Let now $d \geq 3$ and suppose the lemma holds in dimension $d-1$. To prove the result for dimension d we again consider two cases;
a) At least one coordinate of y is 0. Without loss of generality, we assume that $y_d = 0$. Then, by the inductive hypothesis, there are $2d-2$ disjoint paths from 0 to y of length at most $|y|+6$ contained in the hyperplane $H_d = \{(z_1,\ldots,z_{d-1},0)\}$ and two more disjoint paths of length $|y|+2$ are obtained as follows: for the first of these paths, start with the edge going from the origin to $\mathbf{e}_d$ (the last vector in the canonical basis of $\mathbb{R}^d$), then continue with the edges of a path of length $|y|$ which remains on the hyperplane $H_d+\mathbf{e}_d$ going from $\mathbf{e}_d$ to $y+\mathbf{e}_d$ and finally append the edge going from $y+\mathbf{e}_d$ to y. For the second of these paths start with the edge he going from the origin to $-\mathbf{e}_d$ and proceed similarly.
b) All coordinates of y are nonzero. Let $y = (y_1,\ldots,y_d)$ and let $y' = (y_1,\ldots,y_{d-1},0)$. To simplify the notation we assume without loss of generality that $y_i > 0$ for $1 \leq i \leq d-1$ By the inductive hypothesis, in the hyperplane determined by $x_d = 0$, there are $2d-2$ disjoint paths of length at most $|y'|+6$ going from the origin to y'. Hence there are $2d-2$ disjoint paths contained in the same hyperplane, of length at most $|y'|+5$ from the origin to the points of the set $\{y'+\mathbf{e}_i, y'-\mathbf{e}_i : 1 \leq i \leq d-1\}$. For each $i \in \{1.\ldots,d-1\}$, the path whose end point is $y'+\mathbf{e}_i$ is then concatenated with the path of length y_d starting and that point ending at $y'+\mathbf{e}_i+y_d\mathbf{e}_d$ and with the path of length 1 from $y'+\mathbf{e}_i+y_d\mathbf{e}_d$ to y. Similarly, we concatenate the path ending at $y'-\mathbf{e}_i$ with a path starting at that point and ending at y. The paths thus obtained are disjoint and each of them has length at most $|y'|+5+y_d+1 = |y|+6$. It remains to exhibit two extra paths from the origin to y with the desired properties. The first of these paths is obtained by concatenating the only path of length y_d+1 going from the origin to $(y_d+1)\mathbf{e}_d$, with a path of length $|y'|$ from $(y_d+1)\mathbf{e}_d$ to $y+\mathbf{e}_d$ and with the path of length 1 from $y+\mathbf{e}_d$ to y. The second of these paths is obtained by concatenating the path of length 1 going from the origin to $-\mathbf{e}_d$ with a path of length $|y'|$ from $-\mathbf{e}_d$ to $y'-\mathbf{e}_d$ and with the path of length y_d+1 from $y'-\mathbf{e}_d$ to y. These last two paths have length $|y|+2$ and complete the proof.

□

Lemma 4.2. *Under condition* (1.6)*, we have* $E(t(\mathbf{0},x)^d) < \infty$*, for all* $x \in \mathbb{Z}^d$*.*

Proof. Of course, it suffices to consider $x \neq \mathbf{0}$. By Lemma 4.1 there exist $2d$ disjoint paths $\pi_1,\ldots,\pi_{2d}$ from $\mathbf{0}$ to x whose lengths are at most $|x|+6$.

Thus, we may write for all $s > 0$

$$P(t(\mathbf{0}, x) > s) \leq \prod_{i=1}^{2d} P(t(\pi_i) > s). \tag{4.1}$$

But if the length of π is at most C we have $P(t(\pi_i) > s) \leq CP(\tau(e) > s/C)$ and we have $P(t(\mathbf{0}, x) > s) \leq C^{2d} P(Z > s/C)$, where $Z = \min\{t_1, \ldots, t_{2d}\}$, with $t_1, \ldots, t_{2d}$ i.i.d. distributed according to F. The result follows at once. □

As in Theorem 1.3, we see that for any $x \in \mathbb{Z}^d$

$$\lim_{n\to\infty} \frac{1}{n} t(\mathbf{0}, nx) = \mu(x) := \inf_{n\geq 1} \frac{1}{n} E\left(t(\mathbf{0}, nx)\right) \quad a.s. \text{ as } n \to \infty. \tag{4.2}$$

This can be extended to any direction $x \in \mathbb{Q}^d$. Indeed, let $M \geq 1$ be an integer and let $\mathcal{V}_M = \{x \in \mathbb{R}^d \colon Mx \in \mathbb{Z}^d\}$. For any $x \in \mathcal{V}_M$ we may apply the sub-additive ergodic theorem as before to the variables

$$X_{m,n} = t(mMx, nMx),\ m \leq n$$

to get

$$\lim_{n\to\infty} \frac{1}{nM} t(\mathbf{0}, nMx) = \mu(x) := \inf_{n\geq 1} \frac{1}{nM} E\left((t(\mathbf{0}, nMx)\right) \text{ a.s and in } L^1.$$

It is a simple exercise to show that the definition is well posed, i.e. it does not depend on the choice of the integer M verifying $Mx \in \mathbb{Z}^d$. We easily see that this extends (4.2), defines $\mu(x)$ uniquely for any $x \in \mathbb{Q}^d = \cup_{M\geq 1} \mathcal{V}_M$, and

$$\mu(rx) = r\mu(x) \text{ for all } x \in \mathbb{Q}^d \text{ and } r \geq 0 \text{ rational}. \tag{4.3}$$

Moreover, if $x \in \mathcal{V}_M, y \in \mathcal{V}_N$ then $x - y \in \mathcal{V}_{MN}$ and

$$\mu(y) - \mu(x) = \lim_{n\to\infty} \frac{1}{nMN} (t(\mathbf{0}, nMNy) - t(\mathbf{0}, nMNx)). \tag{4.4}$$

Since

$$t(\mathbf{0}, nMNy) \leq t(\mathbf{0}, nMNx) + t(nMNx, nMNy)$$

and $t(nMNx, nMNy)$ has the same distribution as $t(\mathbf{0}, nMN(y - x))$, we get

$$\mu(y) - \mu(x) \leq \mu(y - x). \tag{4.5}$$

The model is invariant under permutations of the coordinates or reflections on the axis, so that $\mu(x) = \mu(\tilde{x})$ if the $x, \tilde{x} \in \mathbb{Q}^d$ differ by a permutation of the coordinates or their signs. In particular

$$\mu(x_1, \ldots, x_d) \leq \sum_{i=1}^{d} \mu(x_i \mathbf{e}_i) = |x|\mu \tag{4.6}$$

with $\mu = \mu(\mathbf{e}_1)$ the time constant in Theorem 1.3. Thus

$$|\mu(y) - \mu(x)| \leq \mu(y - x) \leq \mu|x - y|, \quad \text{for all } x, y \in \mathbb{Q}^d \tag{4.7}$$

and we may extend $\mu(\cdot)$ by continuity to the whole $\mathbb{R}^d$. This function, that we continue to denote by μ, satisfies (4.6), (4.7), it is invariant under permutations and reflections, and it is homogeneous, i.e.

$$\mu(0) = 0, \mu(ax) = a\mu(x) \text{ for all } a \geq 0. \tag{4.8}$$

Finally, assume that $\mu(x) = 0$ for some $x \neq \mathbf{0}$. Without loss of generality $x_1 \neq 0$, in which case

$$2|x_1|\mu = \mu(2x_1\mathbf{e}_1) \leq \mu(x) + \mu(x_1, -x_2, \ldots, -x_d) = 0 \tag{4.9}$$

where we used the first inequality in (4.7) and the invariance under change of signs of the coordinates. By (4.6) it follows that $\mu(x) = 0$ for all $x \in \mathbb{R}^d$. That is, if the time constant μ does not vanish, $\mu(x)$ defines a norm on $\mathbb{R}^d$.

Let $B_0 = \{x \colon \mu(x) \leq 1\}$. Thus $B_0 = \mathbb{R}^d$ if and only if the time constant μ vanishes. Otherwise B_0 is the closed unit ball for a given norm in $\mathbb{R}^d$. Hence, it has a non-empty interior and is both compact and convex. Moreover, since μ is invariant under permutations of the coordinates and reflections, B_0 shares those properties..

Remark. Considering the definition of $B(t)$ in (1.5), it is also natural to extend the definition of $t(\mathbf{0}, x)$ to each $x \in \mathbb{R}^d$ setting $t(\mathbf{0}, x) = t(\mathbf{0}, \hat{x})$ if $\hat{x} \in \mathbb{Z}^d$ and $x \in \hat{x} + [-1/2, 1/2)^d$. We leave as an exercise to show that under the conditions of Theorem 1.3, the radial limits exist a.s. in all directions and are given by $\mu(x)$:

$$\lim_{n\to\infty} \frac{1}{n} t(\mathbf{0}, nx) = \mu(x) \text{ a.s., for all } x \in \mathbb{R}^d. \tag{4.10}$$

To get the shape theorem requires controlling the regularity in a stronger form, where we need condition (1.6). We now focus on the proof of (1.7) in the case of positive time constant.

We want to prove that

$$P\left((1-\epsilon)B_0 \subset \frac{1}{t}B(t), \; \forall t \text{ large}\right) = 1 \text{ for each } \epsilon > 0. \tag{4.11}$$

and

$$P\left(\frac{1}{t}B(t) \subset (1+\epsilon)B_0 \, \forall t \text{ large}\right) = 1 \text{ for each } \epsilon > 0. \tag{4.12}$$

Proof of (4.11)**.**

Taking into account the compactness of B_0 and the uniform continuity of $\mu(\cdot)$, (4.11) follows if we show that for each x such that $\mu(x) < 1$ we can find $\delta > 0$ so that

$$P\left(D_\delta(x) \subset \frac{1}{t}B(t), \; \forall t \text{ large}\right) = 1. \tag{4.13}$$

where

$$D_\delta(x) := \{y \in \mathbb{R}^d \colon |y - x| < \delta\} \tag{4.14}$$

For $x = \mathbf{0}$ in (4.13), the statement will follow at once (by Borel-Cantelli) if we can prove that

$$\sum_{z\in\mathbb{Z}^d} P(t(\mathbf{0},z) > C|z|) < \infty, \tag{4.15}$$

for some $C < \infty$.

Remark. The validity of (4.15) for some $C < \infty$ would easily follow from Chebychev's inequality had we assumed that $E(\tau(e))^2 < +\infty$, which is a stronger assumption than (1.6) (Exercise). This can be left as a further exercise.

For the proof of (4.15) under condition (1.6), we first consider $z \in (5\mathbb{Z})^d$ and then control the times $t(z,x)$ for $z \in (5Z)^d, x \in z + (-5/2, 5/2]^d$. This is a trick that allows the use of second moment tools.

By the proof of Lemma 4.1, if $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$, we may define $2d$ disjoint paths $\pi_i, 1 \le i \le 2d$ from $5x$ to $5y$ lying in the d dimensional rectangle $\{5x, 5y\} + (-5/2, 5/2]^d$.

For x, y as above we set $\tau(5x, 5y) = \min\{t(\pi_i), 1 \le i \le 2d\}$. It follows from the assumption (1.6) and the proof of Lemma 4.2 that $E\left(\tau(5x,5y)^d\right) < \infty$, and of course $t(5x,5y) \le \tau(5x,5y)$. Thus, if $x \in \mathbb{Z}^d$ and $x \ne \mathbf{0}$, taking any sequence $x_0 = \mathbf{0}, x_1, \ldots, x_k = x$ with $|x_i - x_{i-1}| = 1$, one has

$$t(0,5x) \le \sum_{i=1}^k t(5x_{i-1}, 5x_i) \le \sum_{i=1}^k \tau_i,$$

where $\tau_i := \tau(5x_{i-1}, 5x_i)$. Due to the restriction on the paths, the variables τ_i and τ_j are independent if $|j-i| > 1$ as they depend on disjoint sets of edges. Since these variables are identically distributed and have second moment,

$$Var(\sum_{i=1}^k \tau_i) = \sum_{i=1}^k Var(\tau_i) + \sum_{i=1}^{k-1} 2Cov(\tau_i, \tau_{i+1}) \le 3kVar(\tau_1),$$

which yields an estimate that yet is not enough to sum over all $(5\mathbb{Z})^d$, but will be useful:

$$P\left(\sum_{i=1}^k \tau_i > k(E\tau_1 + 1)\right) \le \frac{3}{k} Var\left(\tau_1\right). \tag{4.16}$$

To improve this estimate we recall that given any $z \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ there are $2d$ disjoint paths $\tilde{\pi}_1, \ldots, \tilde{\pi}_{2d}$ from $\mathbf{0}$ to z, each one with at most $|z| + 6$ edges. Writing $< z^j_{i-1}, z^j_i >, i = 1\ldots, k_j$ for the edges used in the jth path ($z_0 = \mathbf{0}, z_{k_j} = z$), we may write

$$t(0,5z) \le R_0 + R_1 + \min_{1\le j\le 2d} U_j, \tag{4.17}$$

where

$$\begin{aligned} U_j &= \sum_{i=2}^{k_j-1} \tau(5z^j_{i-1}, 5z^j_i) \\ R_0 &= \max_{1\le j\le 2d} \tau(0, 5z^j_1) \\ R_1 &= \max_{1\le j\le 2d} \tau(5z^j_{k_j-1}, 5z). \end{aligned}$$

From the previous estimates we see that

$$P(R_0 > |z|) = P(R_1 > |z|) \le 2dP(\tau(\mathbf{0}, 5\mathbf{e}_1) > |z|)$$

and since $|z| \le k_j \le |z| + 6$ we see that

$$P\left(U_j > (E\tau_1 + 1)(|z| + 4)\right) \le 3\frac{Var(\tau_1)}{|z| + 4}.$$

Now, note that for $|z| \ge n_0 := 4(E\tau_1 + 1)$ we have:

$$(E\tau_1 + 2)|z| \ge (E\tau_1 + 1)(|z| + 4).$$

Hence, using (4.17) for $|z| \ge n_0$, we have:

$$\begin{aligned} P\left(t(\mathbf{0}, 5z) > (E\tau_1 + 4)|z|\right) &\le P\left(R_0 \ge |z|\right) + P\left(R_1 \ge |z|\right) \\ &\quad + P\left(\min_{1\le j\le 2d} U_j \ge (E\tau_1 + 1)(|z| + 4)\right). \end{aligned}$$

Therefore

$$P(t(\mathbf{0}, 5z) > (E\tau_1 + 4)|z|) \le (3Var(\tau_1))^{2d}|z|^{-2d} + 4dP(\tau_1 > |z|). \tag{4.18}$$

This is summable over $\mathbb{Z}^d$, but it remains to control $x \in \mathbb{Z}^d \setminus (5Z)^d$. If $x \in 5z + (-5/2, 5/2]^d$, we use

$$t(\mathbf{0}, x) \le t(\mathbf{0}, 5z) + t(5z, x) \le t(\mathbf{0}, 5z) + \sup_{y\in 5z+(-5/2,5/2]^d} t(5z, y). \tag{4.19}$$

Noticing that the $t(5z, x)$ is controlled by the minimum time over $2d$ disjoint paths of uniformly bounded length, we have as before

$$P\left(\sup_{x\in 5z+(-5/2,5/2]^d} t(5z, x) > a\right) \le P(R > a) \quad \text{for all } a \in [0, \infty), \tag{4.20}$$

where R is a random variable whose distribution does not depend on z and is such that $E(R^d) < \infty$. It now follows from (4.19) that

$$\begin{aligned} P\left(t(\mathbf{0}, x) > (E\tau_1 + 6)\frac{|x|}{5}\right) &\le P\left(\sup_{x\in 5z+(-5/2,5/2]^d} t(5z, x) > \frac{|x|}{5}\right) \\ &\quad + P\left(t(\mathbf{0}, 5z) > (E\tau_1 + 5)\frac{|x|}{5}\right). \end{aligned} \tag{4.21}$$

Assume now that $|x|$ is large enough to satisfy $(E\tau_1+4)|z| \le (E\tau_1+5)\frac{|x|}{5})$ and $\frac{|x|}{|z|} \ge \frac{4}{3}$. It then follows from (4.18), (4.19) , (4.20) and (4.21) that

$$P\left(t(\mathbf{0},x) > (E\tau_1+6)|x|/5\right) \le (4Var(\tau_1))^{2d}|x|^{-2d} + 4dP(\tau_1 > |x|/5) + P(R > |x|/5). \tag{4.22}$$

Hence, (4.15) follows with $C = (E\tau_1+6)/5$.

We now consider the case when $x \neq \mathbf{0}$, but $\mu(x) < 1$ and prove (4.13) for suitably small δ. Let $A_x := \cup_{t\ge 0}(tD_\delta(x)) = \cup_{t\ge 0}D_{\delta t}(tx)$. We see at once that (4.13) is equivalent to the statement that with probability one, for all but finitely many $z \in \mathbb{Z}^d \cap A_x$ we have

$$t(\mathbf{0},z) \le \inf\{t\colon z \in D_{\delta t}(tx)\} =: s_z.$$

Given x as above, we pick ϵ such that $\mu(x) < 1-\epsilon$ and recall that from (4.10) (the a.s. existence of radial limits in the direction x), we have

$$P(t(\mathbf{0},s_z x) > (\mu(x)+\epsilon)s_z \text{ i.o.}) = 0,$$

where i.o. means for all but a finite number of z's in $\mathbb{Z}^d \cap A_x$.

Recalling that $t(\mathbf{0},z) \le t(\mathbf{0},s_z x) + t(s_z x, z)$, we need to estimate $t(s_z x, z)$. Since $|z - s_z x| = \delta s_z$, if we take δ so that $(1-\mu(x)-\epsilon)/\delta > C)$, with C as in (4.15) we estimate $P(t(s_z x, z) > (1-\mu(x)-\epsilon)s_x) \le P(t(\mathbf{0}, z - s_z x) > C|z - s_z x|$ as before to conclude the proof of (4.11).

Proof of (4.12)**.**

It suffices to show that if $\mu(x) > 1$, there exists a $\delta > 0$ so that

$$P\left(\frac{1}{t}B(t) \cap D_\delta(x) = \emptyset, \quad \text{for all } t \text{ large}\right) = 1$$

We take $\epsilon > 0$ such that $\mu(x) - 1 - 2\epsilon > 0$ and δ so that $(\mu(x)-1-2\epsilon)/\delta > C$, with C in the previous part of the proof. One gets

$$P(t(\mathbf{0},s_z x) < (\mu(x)-\epsilon)s_z \text{i.o.}) = 0$$

and

$$P(t(s_z x, z) > (\mu(x)-1-2\epsilon)s_z \text{i.o.}) = 0$$

Thus

$$P(t(\mathbf{0},z) \le (1+\epsilon)s_z \text{ i. o.}) = 0$$

If δ is chosen small enough, then $(1+\epsilon)s_z \ge s'_z := \sup\{t\colon |z - xt| < \delta t\}$ for all $|z|$ sufficiently large. This shows that with probability one, except possibly for finitely many $z \in A_x$ we have $z \notin B(t)$ whenever $z \in D_{\delta t}(tx)$, therefore completing the proof of (4.12).

The case $B_0 = \mathbb{R}^d$ (Part b) of the theorem) is much easier. The proof follows easily from what we have done in the proof of (4.11).

It remains to prove that if condition (1.6) fails, then (1.9) holds. The argument is very similar to that used in Remark 3.1. Indeed, if $z \in \mathbb{Z}^d$, $z \neq \mathbf{0}$,

$$t(\mathbf{0},z) \ge \min\{\tau(e)\colon e \text{ incident to } z\} =: Y(z).$$

For each z, $Y(z)$ has the same distribution as that $\min\{t_1, \dots, t_{2d}\}$, with the t_i i.i.d. with distribution F and the random variables $\{Y(z)\colon z \in (2\mathbb{Z})^d\}$ (all coordinates of z are even numbers) are all independent. On the other hand, for each $a > 0$,

$$\sum_{z \in (2\mathbb{Z})^d} P(Y(z) \geq a|z|) = \sum_{k=0}^{\infty} \sum_{z \in (2\mathbb{Z})^d} \mathbf{1}_{\{|z|=2k\}} P(\min\{t_1, \dots, t_{2d}\} \geq 2ak)$$

$$\geq C_1 \sum_{k=0}^{\infty} (k+1)^{d-1} P(\min\{t_1, \dots, t_{2d}\} \geq 2ak),$$

for suitable $C_1 > 0$. We see that this sum diverges if (1.6) does not hold. The conclusion follows from Borel-Cantelli. $\square$

Remark. Another basic quantity in the study of the fpp model is the passage time from a point to a hyperplane:

$$b_{0,n} := t(\mathbf{0}, H_n) = \inf\{t(\mathbf{0}, z)\colon z \in H_n\}$$

where $H_n = \{(n, i_2, \dots, i_d)\colon (i_2, \dots, i_d) \in \mathbb{Z}^{d-1}$ (see (1.3)). From Theorem 1.1 (see [20]) one can also prove that under (1.6)

$$\lim_{n \to \infty} \frac{1}{n} b_{0,n} = \lim_{n \to \infty} \frac{1}{n} t(\mathbf{0}, n\mathbf{e}_1) = \mu \text{ a.s.}$$

5. Inequalities for time constants

In spite of the robust derivation of general properties that involve subadditivity, with the convexity and regularity of the "shape function" $\mu(\cdot)$, not much is know about its precise form for a given distribution F. There is a good deal of qualitative information but we lack examples where the time constant μ_F can be explicitly computed. We may wonder what is known on the comparison of the time constants associated to two different distribution functions F and $\tilde{F}$: if F is close to $\tilde{F}$ in some sense, is μ_F close to $\mu_{\tilde{F}}$? The first natural question regards continuity, which was answered by Cox and Kesten:

Theorem 5.1. (Cox-Kesten [10]**, Kesten** [17]**.)** *If $F_n, n \geq 1$ is a sequence of distribution functions on $[0, +\infty)$ and $F_n \to F$ weakly in $[0, +\infty)$, then $\mu_F = \lim_{n\to\infty} \mu_{F_n}$.*

Another natural question has to do with the comparison of μ_F and $\mu_{\tilde{F}}$ when the distributions F and $\tilde{F}$ are comparable in some natural order, as the usual stochastic domination: F is stochastically larger than $\tilde{F}$ if $F(t) \leq \tilde{F}(t)$ for all t.

Exercise. If F is stochastically larger than $\tilde{F}$, we may construct (e.g. on $\Omega = [0, 1)$ with its usual Borel sigma-algebra) a pair of random variables $(\tilde{X}, X)$ so that X has distribution F and $\tilde{X}$ has distribution $\tilde{F}$ and $\tilde{X}(\omega) \leq X(\omega)$ for each ω. We may take the product measure of this simple coupling, to obtain the two configurations of time variables in first passage percolation in such a way that $\tilde{\tau}(e) \leq \tau(e)$ for all edges e.

From the previous exercise we see at once that if F is stochastically larger than $\tilde{F}$, we must have $\mu_{\tilde{F}} \leq \mu_F$. The interesting thing is that, under conditions, the inequality will be strict unless $F = \tilde{F}$. This was proved by van den Berg and Kesten for $d \geq 2$ in a more general situation, i.e. using a more relaxed order (see Definition 5.3 below). It is clear that the condition $F(0) < p_c$ is needed, since otherwise we have $\mu_F = 0$ (Theorem 1.2). In [7] the following condition is introduced:

Definition 5.2. *A distribution F with support in $[0, +\infty)$ is called useful if the following holds:*

$$F(r) < p_c \text{ when } r = 0, \qquad F(r) < \overrightarrow{p_c} \text{ when } r > 0, \tag{5.1}$$

where p_c ($\overrightarrow{p_c}$) denotes the critical probability for the Bernoulli (oriented, resp.) edge percolation model on $\mathbb{Z}^d$, and r stands for the minimum of the support of F, denoted by supp(F).

The partial order considered in [7] and which fits well to the comparison of time constants is the following:

Definition 5.3. *Let F and $\tilde{F}$ be two distributions on $\mathbb{R}$. We say that $\tilde{F}$ is more variable than F (and denote it by $\tilde{F} \succeq F$) if*

$$\int \phi d\tilde{F} \leq \int \phi dF$$

for every concave increasing function $\phi \colon \mathbb{R} \to \mathbb{R}$ for which the two integrals converge absolutely. If, moreover, the two distributions are distinct, we say that $\tilde{F}$ is strictly more variable than F.

Remark. We immediately see that if F is stochastically larger than $\tilde{F}$, then $\tilde{F}$ is more variable than F.

Using this more general concept, van den Berg and Kesten proved the following important result:

Theorem 5.4. (Berg-Kesten [7].) *(a) Let F and $\tilde{F}$ be distributions on $[0, +\infty)$ with finite mean and such that $\tilde{F}$ is more variable than F. Then $\mu_{\tilde{F}} \leq \mu_F$.*
(b) If in addition F is useful and $F \neq \tilde{F}$, then $\mu_{\tilde{F}} < \mu_F$. [2]

These conditions have been relaxed by Marchand (see [26]):
• The first improvement in [26] regards the removal of the assumption of finite mean in part (a). If $\tilde{F}$ is more variable than F, we also have that $\tilde{F}_T$ is more variable than F_T, where F_T and $\tilde{F}_T$ refer to the corresponding truncated distributions at T (i.e. $F_T(x) = F(x)$ if $x < T$ and $F_T(x) = 1$

[2]Recall that we are assuming $d \geq 2$ throughout.

if $x \geq T$ and similarly for $\tilde{F}_T$). Applying Theorem 5.4 (a) and Cox and Durrett's continuity result, we have have $\mu_{\tilde{F}} \leq \mu_F$.

• The other important improvement, applicable in the two-dimensional case, is the removal of the extra condition in Definition 5.2:

Theorem 5.5. (Marchand [26]**.)** *If $d = 2$, $F(0) < p_c$ and $\tilde{F}$ is strictly more variable than F, then $\mu_{\tilde{F}} < \mu_F$.*

In order to have an idea of what is behind the strict comparison, we may focus in the simpler situation when $\min \operatorname{supp}(F) = 0$, F is stochastically larger than $\tilde{F}$ and $\tilde{F} \neq F$. The basic idea is to take a coupling of the random variables so that $P(\tau(e) \geq \tilde{\tau}(e)) = 1$ and $P(\tau(e) - \eta \geq \tilde{\tau}(e)) > 0$. Since μ_F is the limit (in probability) of $t(\mathbf{0}, n\mathbf{e}_1)/n$ as $n \to \infty$, it would suffice to show that the optimal path π_n contains a number δn disjoint segments π_n^i such that $t(\pi_n^i) - \eta \geq \tilde{t}(\pi_n^i)$, in such a way to have $t(\mathbf{0}, n\mathbf{e}_1) - \delta\eta n \geq \tilde{t}(\pi_n) \geq \tilde{t}(\mathbf{0}, n\mathbf{e}_1)$ with probability that tends to one as n goes to ∞.

The construction in [7] has therefore two main ingredients:

• Suitable joint construction of $(\tilde{\tau}, \tau)$ which will be needed for the construction of the segments mentioned above. This depends on properties such as Lemma 5.9 stated below.

• A renormalization argument which will guarantee that with large probability the optimal paths for F must contain a large number of disjoint segments with the proper conditions, allowing a time reduction when passing to the $\tilde{\tau}$ variables. In the simpler situation that we mentioned above, this depends on the existence of $\epsilon > 0$ so that $F(\epsilon) < p_c$ and estimates on sub-critical percolation. To get a feeling of this fact we state below Lemma (5.5) from [7] that gives an estimate under the condition of F being useful. The situation is more involved when $r = \min \operatorname{supp}(F)$ is positive and is atom of F, in which case it involves oriented percolation with the assumption that the probability attributed to this atom ($P(\tau(e) = r)$) is smaller than the critical parameter for oriented percolation. (This is very natural since otherwise there are arbitrarily long paths with the minimal time.)

Lemma 5.6. *If the distribution F of the time variables is useful, then there exist positive numbers $\delta = \delta(F)$ and $D_0 = D_0(F)$ such that*

$$P(t(u, v) \leq (r + \delta)|u - v|) \leq e^{-D_0|u-v|}, \tag{5.2}$$

for all $u, v \in \mathbb{Z}^d$, where r is as in Definition 5.2.

Remark. Although it is assumed throughout [7] that F has finite first moment, this requirement is not used in the proof of the previous lemma.

The previous lemma is obtained through the use of renormalization techniques (where sites are replaced by cubes of certain side length N). This is quite common in percolation, where the renormalized site (or edge) is said to be "open" or "closed" (or painted black /white) according to some property which depends on what happens inside the cube and on the neighboring cubes). This will be used in Section 7.

Remark 5.7. *In the two-dimensional case, Marchand shows that when $F(r) \geq \overrightarrow{p_c}$ there is an analogue of* (5.2) *when v is not in the (oriented) percolation cone from u. (Due to the symmetry of the model, it suffices to consider u, v with non-negative coordinates and refer to the usual oriented percolation in $\mathbb{Z}^2_+$.) For that she uses large deviation estimates on supercritical oriented percolation and a renormalization scheme similar to that in* [7]*. An important consequence, interesting in itself, is Theorem 1.3 in* [26]*, which treats the shape B_0, determining the flat edge $\{x \in B_0 \colon |x| = r\}$ in terms of the asymptotic speed in oriented percolation with parameter $p = F(r)$, when $F(r) > \overrightarrow{p_c}$. (See* [12] *for basic results on planar oriented percolation.)*

For $N \in \mathbb{N}$ and $l = (l^1, \dots, l^d) \in \mathbb{Z}^d$ consider the following partition of $\mathbb{Z}^d$ by hypercubes, as in [7] called N-cubes.

$$S_l(N) = \{x \in \mathbb{Z}^d \colon Nl^i \leq x^i < Nl^i + N, \forall i\}. \tag{5.3}$$

The cubes are naturally indexed by l, and this indexing is also used to define the distance between two N-cubes. If $C \subset \mathbb{Z}^d$, we use $\mathcal{F}(C)$ to denote the σ-field generated by the variables $\tau(e)$ corresponding to edges e that have both endpoints in the set C.

The following collections of boxes $T_l(N)$, $B_l^{+j}(N)$ and $B_l^{-j}(N)$ will also be useful in the proofs: for $N \in \mathbb{N}$, $l \in \mathbb{Z}^d$,

$$\begin{aligned} T_l(N) &= \{x \in \mathbb{Z}^d \colon Nl^i - N \leq x^i \leq Nl^i + 2N, \forall i\}. \\ B_l^{\pm j}(N) &= T_l(N) \cap T_{l \pm 2\mathbf{e}_j}(N), \quad j = 1, \dots, d, \end{aligned} \tag{5.4}$$

where $\mathbf{e}_j, j = 1, \dots, d$ denote the canonical unitary vectors.

We first recall Lemma (5.2) from [7] (see also [15]) which follows from a Peierls argument:

Lemma 5.8. *If the cubes $S_l(N)$ are colored black or white in a random fashion which is (i) translation invariant; (ii) finite range (i.e. the color of $S_l(N)$ is $\mathcal{F}(\cup_{l'}(S(l', N) \colon |l' - l| \leq c_0)$–measurable for a suitable constant c_0) and moreover, $P(S_0(N)$ is black$) \to 1$ as $N \to \infty$, then for all N sufficiently large we can find positive numbers $\epsilon = \epsilon(N)$ and $D = D(N)$ so that for each $u, v \in \mathbb{Z}^d$ the probability that each path from u to v visits at least $\epsilon|u - v|$ distinct black N-cubes is not smaller than $1 - e^{-D|u-v|}$.*

Among several difficulties that the authors have to overcome to pursue their construction in [7], one should stress the role of the condition imposed by Definition 5.2 when $r > 0$ and F has an atom at r. It allows to control the length of segments with the minimal τ value. In [26] and pursuing the proof of Theorem 5.5 this is controlled without this extra assumption, making clever use of large deviation estimates in the two dimensional case. (See [5] for further discussion.)

About the partial order $\succeq$. Two results. Examples

• As mentioned before the basic coupling tool used in [7] and [26] is given by the following

Lemma 5.9. *Let F and $\tilde{F}$ be two distributions on $\mathbb{R}^+$. $\tilde{F} \succeq F$ if and only if there exist a pair of random variables $(\tau, \tilde{\tau})$ on the same probability space, with marginals F and $\tilde{F}$, respectively, and satisfying*

$$E(\tilde{\tau}|\tau = y) \leq y \qquad F\text{- a.s.} \quad y.$$

This result is stated in [7] for measures on $\mathbb{R}$ under the extra condition of finite means, and it follows from a general coupling result in Strassen [31]). The extension is proved in [26].

• A very useful criterion (cut criterion of Karlin and Novikoff):
Let F and $\tilde{F}$ with finite mean. Assume

$$\int x d\tilde{F}(x) \leq \int x dF(x) \tag{5.5}$$

and that for some number z one has:

$$F(x) \leq \tilde{F}(x) \text{ if } x < z;$$
$$F(x) \geq \tilde{F}(x) \text{ if } x > z.$$

Then $\tilde{F} \succeq F$.
The above mentioned results will also be used in Section 7 while discussing a related question.
• We list a few examples related to the order $\succeq$ in relation to the content of this section. These were taken from [7], which might be consulted for some of the details.

(1) If F is the uniform distribution on $[a, b]$ and $\tilde{F}$ the uniform distribution on $[a - \epsilon_2, b + \epsilon_1]$ where $0 < \epsilon_1 \leq \epsilon_2$ so that (5.5) holds, then $\tilde{F} \succeq F$.
(2) If $0 \leq a < b < \infty$, supp$(F) \subset [a, b]$, supp$(\tilde{F}) \subset \{a, b\}$, and (5.5) holds, then $\tilde{F} \succeq F$.
(3) For $a < b$ let $U[a, b]$ be the uniform on the interval $[a, b]$. For integers $0 \leq l \leq m$, let $U\{l, \dots, , m\}$ be the uniform on the set $\{l, \dots, m\}$. Applying Theorem 5.4 one sees that if $d = 2$ and $1 \leq l < m$ then $\mu_{U[l-1/2,m+1/2]} < \mu_{U\{l,\dots,m\}} < \mu_{U[l,m]}$, where as before μ_F is the time constant corresponding to the distribution F
(4) In the situation of the previous example, and from the continuity of the function $F \mapsto \mu_F$, there must exist $c \in (0, 1/2)$ so that $\mu_{U[l-c,m+c]} = \mu_{U\{l,\dots,m\}}$.

Exercise. Find a pair of distributions F and $\tilde{F}$ so that $0 < \mu_{\tilde{F}} < \mu_F$ on $\mathbb{Z}^2$ but $\int x d\tilde{F}(x) > \int x dF(x)$. (See [7] and [30].)

6. Some examples and comments

In this short section we briefly discuss a few questions and results related to the asymptotic shape. There are plenty of open (and mostly hard) questions which are also simple to state. We refer to [5] for a very complete updated discussion. This is just an introduction, and we mention only a few of the most basic questions.

Here one of the first questions that one may ask: which compact convex sets may appear as B_0 in the setting of Theorem 1.1 (assuming of course that $F(0) < p_c$) ? From the basic properties, one knows that B_0 is also symmetric with respect to the coordinate axes and it is not hard to see that that if μ denotes the time constant, then

$$\{x \in \mathbb{R}^d : |x| \leq 1/\mu\} \subset B_0 \subset [-1/\mu, 1/\mu]^d. \tag{6.1}$$

One may wonder if there is any example, e.g. the Eden model, for which B_0 could be an Euclidean ball. Kesten [20] showed this to be false for the Eden model and large values of d: it takes longer to reach $n\mathbf{e}_1$ than $n/\sqrt{d}(1, \dots, 1)$. This proof is based on the behavior of the time constant for large d, when F is the exponential distribution, obtained by Dhar [11]. It was extended in [8] to all values $d \geq 35$. But the question remains open for smaller values of d. One may even wonder if B_0 could be contained in an Euclidean ball of radius $1/\mu$.(See for instance [5] for references).

Even more surprising might be the following: It is believed that a cube in $\mathbb{R}^d$ cannot occur in the context of Theorem 1.1, but this has not been proven. For instance, for $d = 2$ the following question is open (Question 8 at the end of Section 2.5 in [5]): is it impossible for the boundary B_0 in direction $\mathbf{e}_1$ to contain a segment parallel to the $\mathbf{e}_2$?

For $d = 2$, Durrett and Liggett [13] considered a version of Richardson's model with F that has an atom at the minimum of its support (r in the previous section), and showed that if $F(r)$ is larger than the critical value of oriented percolation in $\mathbb{Z}^2$, then the boundary of B_0 on the first quadrant contains a segment of $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = r\}$ i.e. a *flat edge*. To be precise, they considered the situation when the passage times are attached to the vertices of $\mathbb{Z}^2$, but it holds in the same way in the case of edges (see also [17]). This was extended and improved by Marchand in [26], which gave the exact extreme points of this flat edge, as mentioned in Remark 5.7 in Section 5. Following this, in [4] the authors proved the differentiability of boundary of B_0 at the endpoints of the flat edge determined in [26].

Theorems 1.1 and 1.3 represent laws of large numbers. It would be natural to ask what is known about fluctuations. What does one know about the fluctuations of $t(\mathbf{0}, n\mathbf{e}_1)$? Are there available estimates on its variance?

There are conjectures coming from physics that state the existence of an exponent χ depending on the dimension so that

$$Var(t(\mathbf{0}, n\mathbf{e}_1)) \approx n^{2\chi},$$

where for $d = 2$ the value $\chi = 1/3$ is predicted. Kesten ([19]) proved that for any $d \geq 2$, if F has finite second moment and $F(0) < p_c(d)$ then there exists positive constants c, C so that

$$c \leq Var(t(\mathbf{0}, n\mathbf{e}_1)) \leq Cn. \tag{6.2}$$

A first sublinear upper estimate was obtained by Benjamini-Kalai-Schramm [6] when the passage times take two values $0 < a < b < 1$ with probability $1/2$ each, giving an upper bound $Cn/\log n$. This has been extended by various authors (see [5] for the precise references) under suitable conditions. Regarding lower bounds, the improvement of (6.2) obtained by Newman and Pisa [28] gives, under conditions and for $d = 2$, a lower bound of the order $C \log n$. Another important problem regards the difference between $Et(\mathbf{0}, n\mathbf{e}_1)$ and $n\mu$. A very important contribution here comes from the method developed by Alexander [2].

There is a huge recent research work around first passage percolation. To finish this discussion with a few keywords for another important problem we just mention the following: let us say $d = 2$, $F(0) < p_c$, and one looks at the set of optimal paths from from $\mathbf{0}$ to say $n\mathbf{e}_1$. What can one say on its distance to the horizontal axis ? Does it behave like a power of n? (See [27, 28, 24].)

7. First passage percolation and escape trajectories

This section / lecture is based on the paper [3] and also related to the seminal paper [7], which was briefly considered in the previous lecture. We may also motivate it through a *game*: two individuals, that we call λ and σ, move on the vertices of the d-dimensional lattice $\mathbb{Z}^d$, and σ wants to escape from λ. They move through the edges $e \in \mathbb{E}^d$ between nearest neighbor sites. Each edge e has a passage time $\tau(e)$ just as in the first passage percolation model; it represents the time an individual spends in an endpoint of an edge e before moving to the other endpoint of that edge. As before, we assume these times to be i.i.d. random variables and, as in the previous section, we suppose the common distribution F to be *useful*, according to Definition 5.1. We assume λ and σ start from two different positions x_λ and x_σ and ask whether, knowing all the passages times, could σ possibly plan a *perfect strategy*, that is, a strategy that would allow him / her to escape forever, regardless of λ's strategy. If the support of F is bounded, and the clairvoyant player σ can choose the initial position depending on x_λ and the τ variables, then he/she can implement a perfect strategy with probability one. This is stated in part *(i)* of Theorem 7.2, and it is indeed a quite easy fact to be discussed below. On the other hand, in the case F has unbounded support, the answer is frustrating for σ, as we shall see. The main point for the unbounded case is contained in Proposition 7.3, which is indeed the main technical result of [3]. The proof depends on techniques used in [7]. The arguments also allow to obtain Theorem 7.9 which yields further information on the geodesics and the comparison of first passage percolation models, treated in [7] (in a much more general case) in terms of time constants.

A finite path $\pi = (e_1, \dots, e_k)$ is, as before, a sequence of adjacent edges (sharing a vertex), i.e. $e_i =< x_{i-1}, x_i >$ for each $i = 1, \dots, k$. In this case we say that π goes from x_0 to x_k. As in first passage percolation, it suffices to consider self-avoiding paths, i.e. when the sites $x_i, i = 0, \dots, k$ are all distinct, and we always assume this without further comment. Sometimes we identify a path with the sequence of its visited vertices, writing $\pi = (x_0, \dots, x_k)$.

As before, the basic random object consists of a family $\{\tau(e) \colon e \in \mathbb{E}^d\}$ of i.i.d. non-negative random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and where $\tau(e)$ represents the passage time at the edge e, interpreted as the time to traverse e. Their common distribution will be denoted by F. The passage time $t(\pi)$ of a given path $\pi = (e_1, \dots, e_k)$ is simply given by the sum of the variables $\tau(e_i)$ for $i = 1, \dots, k$.

Definition 7.1. *We say that a given path $\tilde{\pi}$ from x to y is optimal (from x to y) if its travel time is the shortest among all paths from x to y:*

$$t(\tilde{\pi}) = \inf\{t(\pi) \colon \pi \text{ is a path from } x \text{ to } y\} = t(x, y),$$

as in (1.2). *Any such optimal path is also called a geodesic (from x to y). An infinite path $\tilde{\pi} = (e_1, e_2, \dots)$ starting at x is said to be a semi-infinite geodesic if for any n the finite path $(e_1, \dots, e_n)$ is a geodesic from x to its endpoint.*

Exercise. Assume that $F(0) < p_c$.
(a) Prove that for any pair $x, y \in \mathbb{Z}^d$ there exists a geodesic from x to y.
(b) Semi-infinite geodesics starting from any given point always exist.
Suggestion: Use part (a) to conclude that given any x there are infinitely many finite geodesics starting from x. Then, conclude that x has a neighbor y such that infinitely many of these geodesics start with the edge with extremities x and y.
(c) When F is continuous, there is a.s. a unique geodesic from x to y for any two distinct vertices x and y.

Theorem 7.2. *Let F be useful in the sense of Definition 5.2.*
(i) Assume F to be supported in $[0, M]$ for some finite M. Let $\tilde{\pi}$ be a semi-infinite geodesic from x_λ. If the event

$$[M + t(x_\sigma, x) < t(x_\lambda, x) \text{ for some } x \in \tilde{\pi}] \tag{7.1}$$

occurs, then σ has a perfect strategy. In particular, given x_λ, with probability one there exist (infinitely many) random initial positions x_σ from where σ has a perfect strategy.
(ii) If F has unbounded support, then $P(\sigma$ has perfect strategy $) = 0$, for any choice of x_λ, x_σ.

Proof of part *(i)***.** (The easy part) Under the situation described in (7.1), it follows at once that a perfect strategy for σ consists in taking any $x \in \tilde{\pi}$ for which $M + t(x_\sigma, x) < t(x_\lambda, x)$, moving to x by the geodesics from x_σ to x and then following the infinite branch of $\tilde{\pi}$ that starts in x. On the other hand, if F is useful it follows at once from the definitions that there exists $\delta > 0$ so that $F(\delta) < p_c$, which implies $t(x_\lambda, x) \to \infty$ as $x \to \infty$ along $\tilde{\pi}$, and the inequality in (7.1) becomes trivial for $x_\sigma \in \tilde{\pi}$ with $t(x_\lambda, x_\sigma) > M$. Part *(ii)* will be proven as a corollary of the next Proposition.

Proposition 7.3. *Let F be a useful distribution on $[0,\infty)$ with unbounded support. For each $M > 0$ let*

(7.2)
$$\bar{t}_{(M)}(\mathbf{0}, x) = \inf\{t(\pi) \colon \pi \textit{ is a path from } 0 \textit{ to } x \textit{ with } \tau(e) \le M \textit{ for all } e \textit{ in } \pi\},$$

with the understanding that $\inf \emptyset = +\infty$*. Then, for each M positive there exists $\epsilon = \epsilon(M) > 0$ and $n_0 = n_0(M)$ so that for all $n \ge n_0$ and all x such that $|x| = n$, we have*

$$P\left(M + t(\mathbf{0}, x) < \bar{t}_{(M)}(\mathbf{0}, x)\right) \ge 1 - e^{-\epsilon n}. \tag{7.3}$$

The proof of the above proposition uses arguments from [7] that were briefly recalled in Section 5. Before getting into the details, let us state the following immediate corollary:

Corollary 7.4. *Let F be a useful distribution on $[0,\infty)$ with unbounded support. Then, for each M positive there exists $\epsilon = \epsilon(M) > 0$ so that for all $n \ge 1$ and all x with $|x| = n$, we have*

(7.4)
$$P(\exists \textit{ geodesic } \pi \textit{ from } \mathbf{0} \textit{ to } x \textit{ such that } \tau(e) \le M \textit{ for all } e \in \pi) \le e^{-\epsilon n}.$$

It is clear that while proving the proposition it suffices to consider $M > 0$ large and such that $P(\tau(e) \in (M, M+1]) > 0$. In the proof we shall also consider optimal paths for the passage times

$$\bar{\tau}(e) = \begin{cases} \tau(e) & \text{if } \tau(e) \le M, \\ +\infty & \text{otherwise.} \end{cases}$$

Black cubes. Recall the notation (5.3) and (5.4) and Lemma 5.6 from the previous section. We then take $\delta = \delta(F)$ and $D_0 = D_0(F)$ so that (5.2) holds. Let $M \in (0, +\infty)$. We now say that the N-cube $S_l(N)$ is *black* if for any path π lying entirely in $T_l(N)$ with endpoints u, v such that $|u-v| \ge N/4$ and using only edges with passage times less than or equal to M, we do have $t(\pi) \ge (r+\delta)|u-v|$. The N-cubes $S_l(N), S_{l'}(N)$ are said to be separated if $T_l(N) \cap T_{l'}(N) = \emptyset$.

From (5.2) we see that Lemma 5.8 applies. In particular, having fixed δ as above, for any N sufficiently large we can take $D = D(N, F) > 0$ and $\epsilon = \epsilon(N, F) > 0$ in a way that for all n large enough:

(7.5)

$$P\{\exists \text{ path from } \mathbf{0} \text{ to } \Gamma_n \text{ that visits at most } [\epsilon n] \text{ separated black } N-\text{cubes}\} \leq e^{-Dn},$$

where $\Gamma_n = \{x \in \mathbb{Z}^d : |x| = n\}$ and $[\cdot]$ denotes the integer part. (Of course, changing D we may assume (7.5) holds for all n.)

We shall now work on the complement of the event on the l.h.s. of (7.5). For the proof of Proposition 7.3 we will try to improve over the optimal paths for $\bar{\tau}$ from 0 to some x in Γ_n by examining the probability of successful shortcuts in disjoint boxes $B_l^{\pm j}(N)$. The main point is the control of the conditional probability of a successful shortcut.

Notation: If π is a path that contains a stretch starting at u and ending at v, we write $\pi_{[u,v]}$ to denote this stretch.

Definition 7.5. *(a) We say that a path π crosses the box $B_l^{\pm j}(N)$ if it crosses the box in the shortest direction and, except for its endpoints, is entirely contained in the interior of $B_l^{\pm j}(N)$.*
(b) We say that a stretch $\pi_{[u,v]}$ of π is shortcutable if it crosses one of the boxes $B_l^{\pm j}(N)$ corresponding to a black cube $S_l(N)$.
(c) For $\rho > 0$ and small, we say that a path from the origin to Γ_n satisfies property $\mathcal{P}_n(\rho)$ if it contains at least $[\rho n]$ (integer part of ρn) shortcutable stretches which lie at distance at least $14N$ of each other.

Lemma 7.6. *Let N be large enough for (7.5) to hold. There exist constants $\rho = \rho(N, F) > 0$ and $D = D(N, F) > 0$ such that for all n the probability that all paths from the origin to Γ_n satisfy condition $\mathcal{P}_n(\rho)$ is at least $1 - \exp(-Dn)$.*

Proof. It is clear that if π is a path connecting a vertex x in $S_l(N)$ to $y \notin T_l(N)$, it must contain a path that crosses one of the $2d$ N–boxes $B_l^{\pm j}(N)$ in the sense just defined. Hence the lemma follows at once from (7.5). $\square$

Construction of Shortcuts

Let π be a path from $\mathbf{0}$ to a point in Γ_n whose edges have passage times less than or equal to M. Let π' be a shortcutable stretch of π and call B the N–box (corresponding to a black cube) it crosses. Assuming π to be optimal for the $\bar{\tau}$ variables, we shall examine the possibility of a successful shortcut $\tilde{\pi}$ for π that uses an edge with passage time larger than M. This would be a path verifying the following conditions:

- $\tilde{\pi}$ and π are edge disjoint;
- the endpoints of $\tilde{\pi}$ coincide with those of a segment π'' of π;
- $|\tilde{\pi}| \leq c_d N$ where the positive constant c_d depends only on the dimension;

- $\tilde{\pi}$ is contained in the same N-box B as π'.
  We shall then say that a shortcut as above is *successful* if $M + t(\tilde{\pi}) < t(\pi'')$.

Let us first assume for notational simplicity that $d = 2$, $B = B^1_l(N)$, which we write as $B = [a, a+N] \times [b, b+3N]$, and that π' crosses B from left to right. Writing $\pi = (x_0, \dots, x_s)$, let $v = x_j$ be the position in $\{a+N\} \times [b+1, b+3N-1]$ where π first reaches the rightmost face of B after entering B and $u = x_i$ the position in $\{a\} \times [b+1, b+3N-1]$ of the leftmost face of B last visited before getting to v, so that $i < j$ and π' is the segment of π that goes from u to v, which we denote as $\pi_{[u,v]}$. We choose $N = 4K$ for some $K \in \mathbb{N}$. We may define as well the vertex with lowest second coordinate and first coordinate in $[a+K, a+3K]$ along $\pi_{[u,v]}$. If there are several such points, let us take e.g. the leftmost one, call it $z = (z^1, z^2)$. We assume that z is on the leftmost half of B, i.e. $z^1 \le a + N/2$ (the argument being analogous when z is on the rightmost half of B); we now define $\tilde{\pi}$ by starting from z moving downwards one step to $z' = z - \mathbf{e}_2$ and then moving horizontally to the right for at most K steps or until we reach any point w in π, whatever comes earlier (note that we can have $w = z'$). In the first case, we then move vertically upwards until reaching a vertex w visited by π; this just defined path from z to w is what we call $\tilde{\pi}$. Three cases have to be analyzed:

(a) $w \in \pi_{[u,v]}$,
(b) w is visited by π before u,
(c) w is visited by π after v.

In all of these three cases we define a new path substituting the stretch of π between w and z by $\tilde{\pi}$

In case (a) the substituted part is the stretch $\pi_{[z,w]}$ contained in $\pi_{[u,v]}$. In this case $|z - w| \ge K$.

In case (b) the substituted part is the portion of π going from w to z.

In case (c) the substituted part is the portion of π going from z to w.

It is easy to check that in all three cases the substituted part of π contains a stretch of $\pi_{[u,v]}$ connecting two points at distance at least $K = N/4$. If $\pi_{[u,v]}$ is shortcutable, then the time of the substituted part is at least $|z-w|r + K\delta$.

The extension to higher dimension is simple and we always have $|z - w| \le 3Nd = 12Kd$.

Assuming that the stretch $\pi_{[u,v]}$ is shortcutable, a condition which guarantees a successful shortcut is $M + \sum_{e_i \in \tilde{\pi}} \tau(e_i) < |z - w|r + K\delta$. Note that the number of edges in $\tilde{\pi}$ is at most $|z - w| + 2$. For our application below (proof of Proposition 7.2), we shall impose for one of the edges, call it e_1, that $\tau(e_1) \in (M, M+1)$ and for the other edges we impose passage times in the interval $[r, r + \delta')$ with $\delta' = \delta/(24d)$. Hence, the shortcut is successful if

$$2M + 1 + (|z - w| + 1)(r + \delta') < |z - w|r + K\delta,$$

which is implied by

$$2M + 1 + r + \delta/(24d) + K\delta/2 < K\delta, \tag{7.6}$$

and this, in its turn, is satisfied for K large enough (depending on M, δ and r).

Proof of Proposition 7.3.

Let δ be as in the definition of black cubes and take $M > 0$ so that $P(\tau \in (M, M+1]) > 0$. We fix $N = 4K$ large so that the conclusion of Lemma 7.6 holds and moreover $2M + 1 + r + \delta/(24d) < K\delta/2$ as in the above construction. We may as well assume that the set on the right side of (7.2) is not empty, and let Π be a path where the minimum is attained. (In case of non-uniqueness, the argument will apply to any of the finitely many optimal paths, and any deterministic way to list them will do the job.)

We now define random variables $U_1, V_1, U_2, V_2, \dots$ taking values in $\mathbb{Z}^d \cup \{\infty\}$. On the event $\{\Pi = \pi\}$, U_1, V_1 are such that $\pi_{[U_1,V_1]}$ is the first shortcutable stretch of π. If no such stretch exists then $U_1 = V_1 = \infty$; U_2, V_2 are such that $\pi_{[U_2,V_2]}$ is the first shortcutable stretch of π after V_1 whose distance to $\pi_{[U_1,V_1]}$ is at least $7N$. In general, U_{i+1}, V_{i+1} is such that $\pi_{[U_{i+1},V_{i+1}]}$ is the first shortcutable stretch of Π after V_i whose distance to $\cup_{j=1}^{i} \Pi_{[U_j,V_j]}$ is at least $7N$, or $U_{i+1} = V_{i+1} = \infty$ if no such stretch exists.

For a given n let $q = q(n) = [\rho n]$. Then, partition the probability space in events as $A(\pi, x_1, y_1, \dots, x_q, y_q) = \{\Pi = \pi, U_i = x_i, V_i = y_i : 1 \le i \le q\}$ and the event $G = \{U_q = +\infty\}$. For each of the shortcutable stretches of π there is a path $\tilde{\pi}_i$ as defined above, with z_i, w_i the corresponding vertices in that construction.

Call $e_{i,1}, \dots, e_{i,k_i}$ the edges of $\tilde{\pi}_i$ and call $e'_{i,1}, \dots, e'_{i,\ell_i}$ the edges which have one endpoint in $\tilde{\pi}_i \setminus \{w_i, z_i\}$ and whose other endpoint is not in $\tilde{\pi}_i$. We now define the event

$$\begin{aligned} F_i(\pi, x_1, y_1, \dots, x_q, y_q) = A(\pi, x_1, y_1, \dots, x_q, y_q) \cap \\ \{\tau(e_{i,1}) \in (M, M+1], \\ \tau(e_{i,2}) < r + \delta', \dots, \tau(e_{i,k_i}) < r + \delta', \tau(e'_{i,1}) > M, \dots, \tau(e'_{i,\ell_i}) > M\}, \end{aligned} \tag{7.7}$$

with $\delta' = \delta/(24d)$ as before. Notice that k_i, ℓ_i are uniformly (in i) bounded by a constant that depends only on K and d.

If the event $F_i(\pi, x_1, y_1, \dots, x_n, y_n)$ occurs then substituting a part of π as explained before we get a new path π'_i, and $M + t(\pi'_i) < t(\pi)$.

Since by Lemma 7.6 $P(G) \le e^{-Dn}$, to conclude the proof it suffices to show that

$$P(\cap_{i=1}^{q} F_i^c(\pi, x_1, y_1, \dots, x_q, y_q) | A(\pi, x_1, y_1, \dots, x_q, y_q)) \le (1 - \varepsilon)^q \tag{7.8}$$

for some $\varepsilon > 0$ (independent of q).

The proof of (7.8) will follow by a suitable application of the following simple lemma.

Lemma 7.7. *Let $\Omega = \mathbb{R}^{\Lambda}$, where Λ is a finite or countable set, endowed with the usual product Borel sigma-field $\sigma(\Lambda)$. Let Λ_1 be a (non-empty) finite proper subset of Λ and $\Lambda_2 = \Lambda \setminus \Lambda_1$. For $i = 1, 2$ let $\Omega_i = \mathbb{R}^{\Lambda_i}$, so that $\Omega = \Omega_1 \times \Omega_2$ and $\sigma(\Lambda) = \sigma(\Lambda_1) \times \sigma(\Lambda_2)$. Let μ_i be a Borel probability measure on $(\Omega_i, \sigma(\Lambda_i))$, $i = 1, 2$ and $\mu = \mu_1 \times \mu_2$ the product measure on $(\Omega, \sigma(\Lambda))$. If $A \in \sigma(\Lambda)$ and $\hat{B} \in \sigma(\Lambda_1)$ have the property that $x = (x_1, x_2) \in A$ and $y_1 \in \hat{B}$ imply $(y_1, x_2) \in A$, then*

$$\mu(B \cap A) \geq \mu(B)\mu(A),$$

where $B = \hat{B} \times \Omega_2$.

Proof. The hypothesis on A and $\hat{B}$ can be written as

$$\mathbf{1}_{\hat{B}}(y_1)\mathbf{1}_A(x_1, x_2) \leq \mathbf{1}_{\hat{B}}(y_1)\mathbf{1}_A(y_1, x_2)$$

for all $x_1, y_1 \in \Omega_1$ and all $x_2 \in \Omega_2$. We compute the iterated integral $\mu_1(dy_1)\mu_1(dx_1)\mu_2(dx_2)$ on both sides. The left hand side yields, by Tonelli's theorem, $\mu(B)\mu(A)$. On the right hand side we have

$$\int_{\Omega_1} \mu_1(dy_1) \int_{\Omega_1} \mu_1(dx_1) \int_{\Omega_2} \mu_2(dx_2) \mathbf{1}_{\hat{B}}(y_1)\mathbf{1}_A(y_1, x_2)$$

which again by Tonelli's theorem can be rewritten as

$$\int_{\Omega} \mu(dy_1, dx_2)\mathbf{1}_{B \cap A}(y_1, x_2) \int_{\Omega_1} \mu_1(dx_1) = \mu(B \cap A)$$

proving the lemma. □

Conclusion of the proof of Proposition 7.3. Let

$$A = A(\pi, x_1, y_1, \ldots, x_q, y_q) = \{\Pi = \pi, U_i = x_i, V_i = y_i : 1 \leq i \leq q\} \text{ and}$$

$$\hat{B}_i = \{\tau(e_{i,1}) \in (M, M+1], \tau(e_{i,2}) < r + \delta', \ldots, \tau(e_{i,k_i}) < r + \delta',$$
$$\tau(e'_{i,1}) > M, \ldots, \tau(e'_{i,\ell_i}) > M\},$$

for $i = 1, \ldots, q$, so that $P(\hat{B}_i) \geq \eta > 0$ for all i. A few instants of reflection show that the condition in the lemma is verified for the pair A and $\hat{B}_1$: since Π is optimal for the $\bar{\tau}$ variables and has a finite time, it cannot cross any of the edges $e'_{1,1}, \ldots, e'_{1,\ell_1}$; this prevents it from using the advantageous edges $e_{1,2}, \ldots, e_{1,k_1}$, and therefore the modified configuration remains in A. Call F_i the event defined in (7.8). Since $F_1 = A \cap \hat{B}_1$, the lemma implies that $P(F_1^c \mid A) \leq 1 - P(\hat{B}_1)$. Analogously, we can again apply the lemma with A replaced by $A \cap \cap_{j=1}^{i-1} F_j^c$ for $i = 2, \ldots, q$ and $\hat{B}_i$ to conclude that conditional probability on the l.h.s. of (7.8) is bounded from above by $(1 - \eta)^{[qn]}$. □

Remark 7.8. *The argument used to prove Proposition 7.3 also shows that under the same conditions, for each M positive there exist $\alpha > 0$ and $\epsilon > 0$ (both depending on M) so that for all $n \geq 1$ and all $x \in \Gamma_n$, we have*

$$P\left(t(\mathbf{0}, x) + \alpha n < \bar{t}_{(M)}(\mathbf{0}, x)\right) \geq 1 - e^{-\epsilon n}. \tag{7.9}$$

Recall now the time constant associated to the passage time distribution F, given by the deterministic limit (in probability):

$$\mu_F = \lim_{n\to\infty} \frac{1}{n} t(\mathbf{0}, n\mathbf{e}_1).$$

As we have seen before, the limit is also a.s. and in L_1 under conditions on the tail of F, e.g. if F has finite mean, and also holds along any fixed direction,

Remark. When F has exponentially decaying tail, the result in Remark 7.8, taking e.g. $x = n\mathbf{e}_1$, can be seen as consequence of large deviation estimates on the variables $t(0,x)/n$. (See e.g.[20].)

Proof of part (ii) of Theorem 7.2.

It is clear that with probability one, no perfect strategy for σ can consist in remaining in a finite set for all times and so it must reach the set $\{x\colon |x - x_\sigma| = n\}$ for any n. It is obvious that on the event $\{t(x_\lambda, x_\sigma) < M\}$ any perfect strategy can only use edges with passage time smaller than M and cannot include finite paths between two points whose passage time exceeds the minimal passage time between these points by more than M. Thus

$$P(\sigma \text{ has perfect strategy}, t(x_\lambda, x_\sigma) < M)$$

$$\leq P(\exists x\colon |x - x_\sigma| = n, \bar{t}_{(M)}(x_\sigma, x) \leq M + t(x_\sigma, x)).$$

Given $\eta > 0$, let M be such that $P(t(x_\lambda, x_\sigma) \geq M) \leq \eta$. Given such M we take n and ϵ so that (7.3) holds and $c_d n^{d-1} e^{-\epsilon n} \leq \eta$, where $c_d n^{d-1}$ is an upper bound for the cardinality of $\{x\colon |x| = n\}$. We then have

$$P(\sigma \text{ has perfect strategy}) \leq 2\eta.$$

$\square$

At this point it is convenient to see that indeed the result can be strengthened without much effort:

Theorem 7.9. *Let F be a useful distribution on $[0,\infty)$ with unbounded support. Then, for each M positive there exists $\epsilon = \epsilon(M) > 0$ and $\alpha = \alpha(M) > 0$ so that for all $n \geq 1$ and all x with $|x| = n$, we have*

(7.10)

$$P\left(\exists \text{ geodesic } \pi \text{ from } \mathbf{0} \text{ to } x \text{ such that } \sum_{e\in\pi} \mathbf{1}_{\{\tau(e)>M\}} \leq \alpha n\right) \leq e^{-\epsilon n}.$$

Proof of Theorem 7.9. An N-cube $S_l(N)$ is now colored black if every geodesic from a point on its boundary to a point on the boundary of $T_l(N)$ uses at least one edge whose passage time is larger than M. From Corollary 7.4 we see that Lemma 5.8 applies, and the result follows. $\square$

Remark. In the statement of Theorem 7.9 one may replace $\{\tau(e) > M\}$ by $\{\tau(e) \in A\}$, where A is any Borel set to which F attributes positive measure. This improves inequality (2.16) in [7].

References

[1] Ahlberg, D. Damron, M. Sidoravicius, V. (2016). Inhomogeneous first-passage percolation. Electron. J. Probab. 21 , no. 4, 119.

[2] Alexander, K. (1997). Approximation of subadditive functions and convergence rates in limiting-shape results. Ann. Probab. bf 1, 30–55.

[3] Andjel, E. D. and Vares, M.E. (2015). First passage percolation and escape strategies. Random Struct. & Algorithms 47, 414423.

[4] Auffinger, A., Damron, M. (2013). Differentiability at the edge of the limit shape and related results in
rst passage percoaltion. Probab. Theory and Rel. Fields **156**, 193–227.

[5] Auffinger, A., Damron, M., and J. Hanson, J. (2016) 50 years of first passage percolation. , arxiv.org/abs/1511.03262

[6] Benjamini, I. Kalai, G. and Schramm, O. (2003). First passage percolation has sublinear distance variance. Ann. Probab. **31**, 1970–1978.

[7] van den Berg, J. and Kesten, H. (1993). Inequalities for the time constant in first-passage percolation. *Ann. Applied Probab.* 3, 56–80.

[8] Couronné, O., Enriquez, N. and Gerin, L. (2011) Construction of a short path in high-dimensional first passage percolation. Elect. Comm. Probab. **16**, 22–28.

[9] Cox, J.T. and Durrett, R. (1981). Some limit theorems for percolation with necessary and sufficient conditions. Ann. Probab. 9, 583603.

[10] Cox, J.T. and Kesten, H. (1981). On the continuity of the time constant of first passage percolation. *J. Appl. Probab.* **18**, 809–819.

[11] Dhar, D. First passage percolation in many dimensions, Phys. Lett. A (1988), **130**, 308310.

[12] Durrett, R. (1984). Oriented percolation in two dimensions. Ann. Probab. **12**, 999–1040.

[13] Durrett, R., Liggett, T. M. (1981) The shape of the limiting set in Richardson's growth model. Ann. Probab. **9**, 186–193.

[14] Eden, M. (1961). A two-dimensional growth model. Proc. Fourth Berkeley Symp. Math. Stat. Probab. **4** J. Neyman (ed), Univ. California Press, 223–239.

[15] Grimmett, G. and Kesten, H. (1984). First passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete* **66** 335–366.

[16] Hammersley, J. M. and Welsh, J. D. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. Springer-Verlag. Bernoulli-Bayes-Laplace Anniversary Volume: 61110.

[17] Kesten, H. (1986). Aspects of first passage percolation. *École d'Eté de Probabilités de Saint-Flour XIV.* Lec. Notes in Math **1180**, 125–264, Springer, New York.

[18] Kesten, H.(1987). Percolation theory and first-passage percolation. Ann. Probab. 15, 12311271, 1987.

[19] Kesten, H. (1993). On the speed of convergence in first-passage percolation. Ann. Appl. Probab. **3**, 296–338.

[20] Kesten, H. (2003) First passage percolation. In: *From classical to modern probability: CIMPA Summer School 2001* . Eds. Pierre Picco, Jaime San Martin. Progress in Probability **54**, 93–143.

[21] Kingman, J. F. C. (1968). The ergodic theory of sub additive stochastic processes. J. Roy. Statist. Soc. Ser. **30**, 499–510.

[22] Kingman, J. F. C. (1973). Subadditive Ergodic Theory. Ann. Probab.**1**, 883–899.

[23] Licea, C. and Newman, C. M. (1996). Geodesics in two-dimensional first-passage percolation, Ann. Probab. 24, 399410, 1996.

[24] Licea, C., Newman, C. and Piza, M. (1996). Superdiffusivity in first-passage percolation. Probab. Th. Rel. Fields **106**, 559–591.

[25] Liggett, T. M. (1985) *Interacting particle systems*, Springer-Verlag.

[26] Marchand, R. (2002) Strict inequalities for the time constant in first passage percolation. *Ann. Appl. Probab.* **12**, 1001–1038.
[27] Newman, C. M. A surface view of first-passage percolation. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1017-1023, Birkhäuser, Basel, 1995
[28] Newman, C. M. and Piza, M. (1995). Divergence of shape fluctuations in two dimensions. Ann. Probab. **23**, 977–1005.
[29] Richardson, D. (1973). Random growth in a tessellation. *Proc. Cambridge Philos. Soc.* **74**, 515–528.
[30] Smythe, R. T. and Wierman, J. C. (1978). First passage percolation on the square lattice. Lec. Notes in Math. **671**, Springer, New York.
[31] Strassen, V. (1965). The existence of probability measures with given marginals, Ann Math Statist. **36**, 423–439.
[32] Wierman, J. C., (1980). Weak moment conditions for time coordinates in firstpassage percolation models. J. Appl. Probab. **17**, 968–978.
[33] Zhang, Y. (2008). Shape fluctuations are different in different directions. Ann. Probab. **36**, 331–362.

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